If a body is moving in any direction, there is a force, arising from the earth's rotation, which always deflects it to the right in the northern hemisphere, and to the left in the southern.

William Ferrel, *The influence of the Earth's rotation upon the relative motion of bodies near its surface,* 1858.

CHAPTER 2

Effects of Rotation and Stratification

THE ATMOSPHERE AND OCEAN are shallow layers of fluid on a sphere in that their thickness or depth is much less than their horizontal extent. Furthermore, their motion is strongly influenced by two effects: rotation and stratification, the latter meaning that there is a mean vertical gradient of (potential) density that is often large compared with its horizontal gradient. Here we consider how the equations of motion are affected by these effects. First, we consider some elementary effects of rotation on a fluid and derive the Coriolis and centrifugal forces, and then we write down the equations of motion appropriate for motion on a sphere. Then we discuss some approximations to the equations of motion that are appropriate for large-scale flow in the ocean and atmosphere, in particular the hydrostatic approximation, and how the presence of strong stratification can be used to further simplify the governing equations.

2.1 THE EQUATIONS OF MOTION IN A ROTATING FRAME OF REFERENCE

Newton's second law of motion, that the acceleration on a body is proportional to the imposed force divided by the body's mass, applies in so-called inertial frames of reference. The earth rotates with a period of about almost 24 hours (23h 56m) relative to the distant stars, and the surface of the earth manifestly is not, in that sense, an inertial frame. Nevertheless, because the surface of the earth is moving (in fact at speeds of up to a few hundreds of meters per second) it is very convenient to describe the flow relative to the earth's surface, rather than in some inertial frame. This necessitates recasting the equations into a form that is appropriate for a rotating frame of reference, and that is the subject of this section.

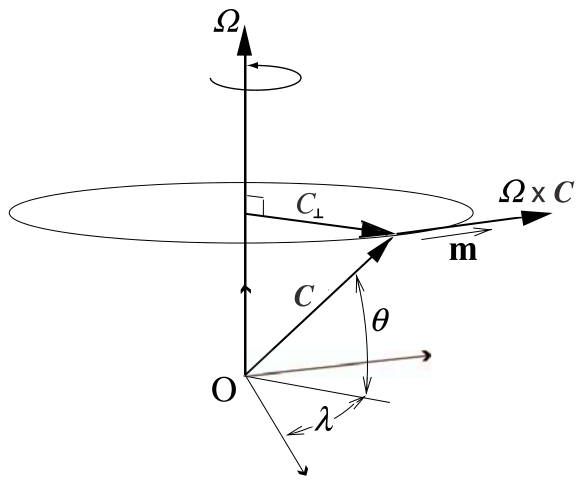

Figure 2.1 A vector $\boldsymbol{C}$ rotating at an angular velocity $\boldsymbol{\Omega}$. It appears to be a constant vector in the rotating frame, whereas in the inertial frame it evolves according to $(\mathrm{d}\boldsymbol{C}/\mathrm{d}t)_I = \boldsymbol{\Omega} \times \boldsymbol{C}$.

2.1.1 Rate of change of a vector

Consider first a vector $\boldsymbol{C}$ of constant length rotating relative to an inertial frame at a constant angular velocity $\boldsymbol{\Omega}$. Then, in a frame rotating with that same angular velocity it appears stationary and constant. If in small interval of time δt it the vector $\boldsymbol{C}$ rotates through a small angle $\delta\lambda$ then the change in $\boldsymbol{C}$, as perceived in the inertial frame, is given by (see Fig. 2.1)

$$\delta \boldsymbol{C} = |\boldsymbol{C}| \cos\theta \, \delta\lambda \, \boldsymbol{m}, \tag{2.1}$$

where the vector $\boldsymbol{m}$ is the unit vector in the direction of change of $\boldsymbol{C}$, which is perpendicular to both $\boldsymbol{C}$ and $\boldsymbol{\Omega}$. But the rate of change of the angle λ is just, by definition, the angular velocity so that $\delta\lambda = |\boldsymbol{\Omega}|\delta t$ and

$$\delta \boldsymbol{C} = |\boldsymbol{C}||\boldsymbol{\Omega}| \sin\hat{\theta} \, \boldsymbol{m} \, \delta t = \boldsymbol{\Omega} \times \boldsymbol{C} \, \delta t. \tag{2.2}$$

using the definition of the vector cross product, where $\hat{\theta} = (\pi/2 - \theta)$ is the angle between $\boldsymbol{\Omega}$ and $\boldsymbol{C}$. Thus

$$\left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I = \boldsymbol{\Omega} \times \boldsymbol{C} \tag{2.3}$$

where the left hand side is the rate of change of $\boldsymbol{C}$ as perceived in the inertial frame.

Now consider a vector $\boldsymbol{B}$ that changes in the inertial frame. In a small time δt the change in $\boldsymbol{B}$ as seen in the rotating frame is related to the change seen in the inertial frame by

$$(\delta \boldsymbol{B})_I = (\delta \boldsymbol{B})_R + (\delta \boldsymbol{B})_{\mathrm{rot}} \tag{2.4}$$

where the terms are, respectively, the change seen in the inertial frame, the change due to the vector itself changing as measured in the rotating frame, and the change due to the rotation. Using (2.2) $(\delta \boldsymbol{B})_{\mathrm{rot}} = \boldsymbol{\Omega} \times \boldsymbol{B} \, \delta t$, and so the rates of change of the vector $\boldsymbol{B}$ in the inertial and rotating frames are related by

$$\boxed{\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{B}}. \tag{2.5}$$

This relation applies to a vector $\boldsymbol{B}$ that, as measured at any one time, is the same in both inertial and rotating frames.

2.1.2 Velocity and acceleration in a rotating frame

The velocity of a body is not measured to be the same in the inertial and rotating frames, so care must be taken when applying (2.5) to velocity. First apply (2.5) to $\boldsymbol{r}$, the position of a particle to obtain

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{r} \tag{2.6}$$

or

$$\boldsymbol{v}_I = \boldsymbol{v}_R + \boldsymbol{\Omega}\times\boldsymbol{r}. \tag{2.7}$$

We refer to $\boldsymbol{v}_R$ and $\boldsymbol{v}_I$ as the relative and inertial velocity, respectively, and (2.7) relates the two. Apply (2.5) again, this time to the velocity $\boldsymbol{v}_R$ to give

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{v}_R, \tag{2.8}$$

or, using (2.7)

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_I - \boldsymbol{\Omega}\times\boldsymbol{r})\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{v}_R, \tag{2.9}$$

or

$$\left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{v}_R + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\times\boldsymbol{r} + \boldsymbol{\Omega}\times\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I. \tag{2.10}$$

Then, noting that

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{r} = (\boldsymbol{v}_R + \boldsymbol{\Omega}\times\boldsymbol{r}), \tag{2.11}$$

and assuming that the rate of rotation is constant, (2.10) becomes

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I - 2\boldsymbol{\Omega}\times\boldsymbol{v}_R - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}). \tag{2.12}$$

This equation may be interpreted as follows. The term on the left-hand side is the rate of change of the relative velocity as measure in the rotating frame. The first term on the right-hand side is the rate of change of the inertial velocity as measured in the inertial frame (or, loosely, the inertial acceleration). Thus, by Newton's second law, it is equal to force on a fluid parcel divided by its mass. The second and third terms on the right-hand side (including the minus signs) are the 'Coriolis force' and the 'centrifugal force' per unit mass. Neither of these are true forces — they may be thought of as quasi-forces (i.e., 'as if' forces); that is, when a body is observed from a rotating frame it seems to behave as if unseen forces are present that affect its motion. If (2.12) is written, as is common, with the terms $+2\boldsymbol{\Omega}\times\boldsymbol{v}_r$ and $+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$ on the left-hand side then these terms should be referred to as the Coriolis and centrifugal *accelerations*.[1]

Centrifugal force

If $\boldsymbol{r}_\perp$ is the perpendicular distance from the axis of rotation (see Fig. 2.1 and substitute $\boldsymbol{r}$ for $\boldsymbol{C}$), then, because $\boldsymbol{\Omega}$ is perpendicular to $\boldsymbol{r}_\perp$, $\boldsymbol{\Omega} \times \boldsymbol{r} = \boldsymbol{\Omega} \times \boldsymbol{r}_\perp$. Then, using the vector identity $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\perp) = (\boldsymbol{\Omega} \cdot \boldsymbol{r}_\perp)\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\boldsymbol{r}_\perp$ and noting that the first term is zero, we see that the centrifugal force per unit mass is just given by

$$\boldsymbol{F}_{\mathrm{ce}} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \Omega^2 \boldsymbol{r}_\perp. \tag{2.13}$$

This may usefully be written as the gradient of a scalar potential,

$$\boldsymbol{F}_{\mathrm{ce}} = -\nabla \Phi_{\mathrm{ce}}. \tag{2.14}$$

where $\Phi_{\mathrm{ce}} = -(\Omega^2 r_\perp^2)/2 = -(\boldsymbol{\Omega} \times \boldsymbol{r}_\perp)^2/2$.

Coriolis force

The Coriolis force per unit mass is:

$$\boldsymbol{F}_{\mathrm{Co}} = -2\boldsymbol{\Omega} \times \boldsymbol{v}_R. \tag{2.15}$$

It plays a central role in much of geophysical fluid dynamics and will be considered extensively later on. For now, we just note three basic properties:

(i) There is no Coriolis force on bodies that are stationary in the rotating frame.

(ii) The Coriolis force acts to deflect moving bodies at right angles to their direction of travel.

(iii) The Coriolis force does no work on a body, a consequence of the fact that $\boldsymbol{v}_R \cdot (\boldsymbol{\Omega} \times \boldsymbol{v}_R) = 0$.

2.1.3 Momentum equation in a rotating frame

Since (2.12) simply relates the accelerations of a particle in the inertial and rotating frames, then in the rotating frame of reference the momentum equation may be written

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla \Phi. \tag{2.16}$$

We have dropped the subscript R; henceforth, unless ambiguity is present, all velocities without a subscript will be considered to be relative to the rotating frame.

2.1.4 Mass and tracer conservation in a rotating frame

Let ϕ be a scalar field that, in the inertial frame, obeys

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} + \phi \nabla \cdot \boldsymbol{v}_I = 0. \tag{2.17}$$

Now, observers in both the rotating and inertial frame measure the same value of ϕ. Further, $\mathrm{D}\phi/\mathrm{D}t$ is simply the rate of change of ϕ associated with a material parcel, and therefore is reference frame invariant. Thus,

$$\left(\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)_R = \left(\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)_I \tag{2.18}$$

where $(\mathrm{D}\phi/\mathrm{D}t)_R = (\partial\phi/\partial t)_R + \boldsymbol{v}_R \cdot \nabla\phi$ and $(\mathrm{D}\phi/\mathrm{D}t)_I = (\partial\phi/\partial t)_I + \boldsymbol{v}_I \cdot \nabla\phi$ and the local temporal derivatives $(\partial\phi/\partial t)_R$ and $(\partial\phi/\partial t)_I$ are evaluated at fixed locations in the rotating and inertial frames, respectively.

Further, since $\boldsymbol{v} = \boldsymbol{v}_I - \boldsymbol{\Omega}\times\boldsymbol{r}$, we have that

$$\nabla \cdot \boldsymbol{v}_I = \nabla \cdot (\boldsymbol{v}_I - \boldsymbol{\Omega}\times\boldsymbol{r}) = \nabla \cdot \boldsymbol{v}_R \tag{2.19}$$

since $\nabla \cdot (\boldsymbol{\Omega}\times\boldsymbol{r}) = 0$. Thus, using (2.18) and (2.19), (2.17) is equivalent to

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} + \phi\nabla \cdot \boldsymbol{v} = 0 \tag{2.20}$$

where all observables are measured in the *rotating* frame. Thus, the equation for the evolution of a scalar whose measured value is the same in rotating and inertial frames is unaltered by the presence of rotation. In particular, the mass conservation equation is unaltered by the presence of rotation.

Although we have taken (2.18) as true *a priori*, the individual components of the material derivative differ in the rotating and inertial frames. In particular

$$\left(\frac{\partial\phi}{\partial t}\right)_I = \left(\frac{\partial\phi}{\partial t}\right)_R - (\boldsymbol{\Omega}\times\boldsymbol{r}) \cdot \nabla\phi \tag{2.21}$$

because $\boldsymbol{\Omega}\times\boldsymbol{r}$ is the velocity, in the inertial frame, of a uniformly rotating body. Similarly,

$$\boldsymbol{v}_I \cdot \nabla\phi = (\boldsymbol{v}_R + \boldsymbol{\Omega}\times\boldsymbol{r}) \cdot \nabla\phi. \tag{2.22}$$

Adding the last two equations reprises and confirms (2.18).

2.2 EQUATIONS OF MOTION IN SPHERICAL COORDINATES

The earth is very nearly spherical and it might appear obvious that we must cast our equations in spherical coordinates. Although this does turn out to be true, the presence of a centrifugal force causes some complications which we must first discuss. The reader who is willing *ab initio* to treat the earth as a perfect sphere and to neglect the horizontal component of the centrifugal force may skip the next section.

2.2.1 * The centrifugal force and spherical coordinates

The centrifugal force is a potential force, like gravity, and so we may therefore define an 'effective gravity' equal to the sum of the true gravity and the centrifugal force. The true gravitational force is directed toward the center of the earth, except possibly for tiny effects due to the earth's lack of sphericity and inhomogeneity, but the line of action of the effective gravity will in general differ slightly from this, and therefore have a component in the 'horizontal' plane, that is the plane perpendicular to the radial direction. The magnitude of the centrifugal force is $\Omega^2 r_\perp$, and so the effective gravity is given by

$$\boldsymbol{g} \equiv \boldsymbol{g}_{\mathrm{eff}} = \boldsymbol{g}_{\mathrm{grav}} + \Omega^2 \boldsymbol{r}_\perp \tag{2.23}$$

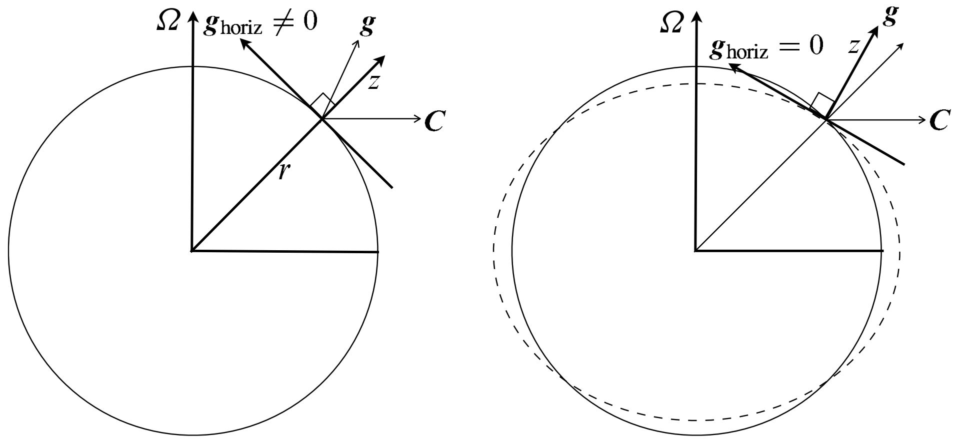

Fig. 2.2 Left: Directions of forces and coordinates in true spherical geometry. $\boldsymbol{g}$ is the apparent gravity (including the centrifugal force, $\boldsymbol{C}$) and its horizontal component is evidently non-zero. Right: a modified coordinate system, in which the vertical direction is defined by the direction of $\boldsymbol{g}$, and so the horizontal component of $\boldsymbol{g}$ is identically zero. The dashed line schematically indicates a surface of constant geopotential. The differences between the direction of $\boldsymbol{g}$ and the direction of the radial coordinate, and between the sphere and the geopotential surface, are much exaggerated and in reality are similar to the thickness of the lines themselves.

where $\boldsymbol{g}_{\text{grav}}$ is the gravitational force due to the gravitational attraction of the earth and $r_\perp = r\cos\vartheta$. Both gravity and centrifugal force are potential forces and therefore we may define the *geopotential,* Φ, such that

$$\boldsymbol{g} = -\nabla\Phi \tag{2.24}$$

Surfaces of constant Φ are not quite spherical because $r_\perp$, and hence the centrifugal force, vary with latitude (Fig. 2.2).

The components of the centrifugal force parallel and perpendicular to the radial direction are $\Omega^2 r\cos^2\vartheta$ and $\Omega^2 r\cos\vartheta\sin\vartheta$. Newtonian gravity is much larger than either of these, and at the earth's surface the ratio of centrifugal to gravitational terms is approximately, and no more than,

$$\alpha \approx \frac{\Omega^2 a}{g} \approx \frac{(7.27\times10^{-5})^2\times6.4\times10^6}{10} \approx 3\times10^{-3} \tag{2.25}$$

(Note that at the equator and pole the horizontal component of the centrifugal force is zero and the effective gravity does point toward the center of the earth.) The angle between $\boldsymbol{g}$ and the line to the center of the earth is given by a similar expression and so is also small, typically around 3×10^{-3} radians. However, the horizontal component of the centrifugal force is still large compared to the Coriolis force, their ratio in mid-latitudes being given by

$$\frac{\text{Horizontal centrifugal force}}{\text{Coriolis force}} \approx \frac{\Omega^2 a\cos\vartheta\sin\vartheta}{2\Omega u} \approx \frac{\Omega a}{4|u|} \approx 10, \tag{2.26}$$

using $u = 10\,\mathrm{m\,s^{-1}}$. The centrifugal term therefore dominates over the Coriolis term, and is largely balanced by a pressure gradient force. Thus, if we adhered to true spherical coordinates, both the horizontal and radial components of the momentum equation would be dominated by a static balance between a pressure gradient and gravity or centrifugal terms. Although in principle there is nothing wrong with writing the equations this way, it obscures the dynamical balances involving the Coriolis force and pressure that determine the large-scale horizontal flow.

A way around this problem is to use the direction of the geopotential force to *define* the vertical direction, and then to regard the surfaces of constant Φ as being true spheres.[2] The horizontal component of apparent gravity is then identically zero, and we have traded a potentially large dynamical error for a very small geometric error. The geopotential Φ is then a function of the vertical coordinate alone, and for many purposes we can just take $\Phi = gz$. In fact, over time, the earth has developed an equatorial bulge to compensate for and neutralize the centrifugal force, so that the effective gravity does in fact act in a direction virtually normal to the earth's surface; that is, the surface of the earth is an oblate spheroid of nearly constant geopotential and, because the oblateness is very small (the polar diameter is about 12,714 km, whereas the equatorial diameter is about 12,756 km) using spherical coordinates is a very accurate way to map the spheroid. The direction normal to geopotential surfaces, the local vertical, is, in this approximation, taken to be the direction of increasing r in spherical coordinates. If the angle between apparent gravity and a natural direction of the coordinate system were not small then more heroic measures would be called for. Note, though, that the equatorial bulge is not a sine qua non of this approximation: if the solid earth were a true sphere the dynamics of the atmosphere would be virtually unaltered, and we would use the same equations to describe that motion, with a similar small geometric error in the coordinates.

2.2.2 Some identities in spherical coordinates

The location of a point is given by the coordinates (λ, ϑ, r) where λ is the angular distance eastward (i.e., longitude), ϑ is angular distance poleward (i.e., latitude) and r is the radial distance from the center of the earth. (See Fig. 2.3. In many fields co-latitude is used as a spherical coordinate, but meteorology and oceanography use latitude.) If a is the radius of the earth, then we also define $z = r - a$. At a given location we may also define the Cartesian increments $(\delta x, \delta y, \delta z) = (r\cos\vartheta\,\delta\lambda, r\delta\vartheta, \delta r)$.

For a scalar quantity ϕ the material derivative in spherical coordinates is

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial\phi}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial\phi}{\partial\lambda} + \frac{v}{r}\frac{\partial\phi}{\partial\vartheta} + w\frac{\partial\phi}{\partial r}, \tag{2.27}$$

where the velocity components corresponding to the coordinates $\lambda, \vartheta, r)$ are

$$(u, v, w) \equiv \left(r\cos\vartheta\frac{\mathrm{D}\lambda}{\mathrm{D}t}, r\frac{\mathrm{D}\vartheta}{\mathrm{D}t}, \frac{\mathrm{D}r}{\mathrm{D}t}\right). \tag{2.28}$$

That is, u is the zonal velocity, v is the meridional velocity and w the vertical

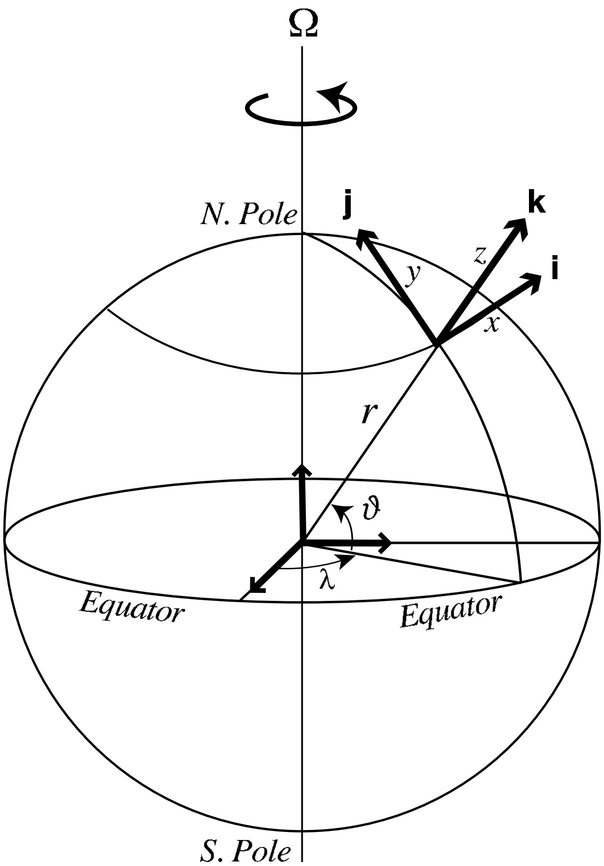

Figure 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.

velocity. If we define $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to be the unit vectors in the direction of increasing (λ, ϑ, r) then

$$\boldsymbol{v} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w. \tag{2.29}$$

Note also that $\mathrm{D}r/\mathrm{D}t = \mathrm{D}z/\mathrm{D}t$.

The divergence of a vector $\boldsymbol{B} = \mathbf{i}B_\lambda + \mathbf{j}B_\vartheta + \mathbf{k}B_z$ is

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\cos\vartheta}\left[\frac{1}{r}\frac{\partial B_\lambda}{\partial \lambda} + \frac{1}{r}\frac{\partial}{\partial \vartheta}(B_\vartheta \cos\vartheta) + \frac{\cos\vartheta}{r^2}\frac{\partial}{\partial r}(r^2 B_r)\right]. \tag{2.30}$$

The vector gradient of a scalar is:

$$\nabla\phi = \mathbf{i}\frac{1}{r\cos\vartheta}\frac{\partial\phi}{\partial\lambda} + \mathbf{j}\frac{1}{r}\frac{\partial\phi}{\partial\vartheta} + \mathbf{k}\frac{\partial\phi}{\partial r} \tag{2.31}$$

The Laplacian of a scalar is:

$$\nabla^2\phi \equiv \nabla\cdot\nabla\phi = \frac{1}{r^2\cos\vartheta}\left[\frac{1}{\cos\vartheta}\frac{\partial^2\phi}{\partial\lambda^2} + \frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \cos\vartheta\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)\right]. \tag{2.32}$$

The curl of a vector is:

$$\operatorname{curl}\boldsymbol{B} = \nabla\times\boldsymbol{B} = \frac{1}{r^2\cos\vartheta}\begin{vmatrix} \mathbf{i}\,r\cos\vartheta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\vartheta & \partial/\partial r \\ B_\lambda r\cos\vartheta & B_\vartheta r & B_r \end{vmatrix} \tag{2.33}$$

The vector Laplacian $\nabla^2 \boldsymbol{B}$ (used for example when calculating viscous terms in the momentum equation) may be obtained from the vector identity:

$$\nabla^2 \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla \times (\nabla \times \boldsymbol{B}) \tag{2.34}$$

Only in Cartesian coordinates does this take the simple form,

$$\nabla^2 \boldsymbol{B} = \frac{\partial^2 \boldsymbol{B}}{\partial x^2} + \frac{\partial^2 \boldsymbol{B}}{\partial y^2} + \frac{\partial^2 \boldsymbol{B}}{\partial z^2}. \tag{2.35}$$

The expansion in spherical coordinates is rather uninformative and rarely needed.

Rate of change of unit vectors

In spherical coordinates the defining unit vectors are $\mathbf{i}$, the unit vector pointing eastward, parallel to a line of latitude; $\mathbf{j}$ is the unit vector pointing polewards, parallel to a meridian; and $\mathbf{k}$, the unit vector pointing radially outward. The directions of these vectors change with location, and in fact this is the case in nearly all coordinate systems, with the notable exception of the Cartesian one, and thus their material derivative is not zero. One way to evaluate this is to consider geometrically how the coordinate axes change with position (2.5). We will approach the problem a little differently, by first obtaining the effective rotation rate $\boldsymbol{\Omega}_{\text{flow}}$, relative to the earth, of a unit vector as it moves with the flow, and then applying (2.3). Specifically, let the fluid velocity be $\boldsymbol{v} = (u, v, w)$. The meridional component, v, produces a displacement $r\delta\vartheta = v\delta t$, and this give rise a local effective vector rotation rate around the local zonal axis of $-(v/r)\mathbf{i}$, the minus sign arising because a displacement in the direction of the north pole is produced by negative rotational displacement around the $\mathbf{i}$ axis. Similarly, the zonal component, u, produces a displacement $\delta\lambda r \cos\vartheta = u\delta t$ and so an effective rotation rate, but now about the earth's rotation axis, of $u/(r\cos\vartheta)$. Now, a rotation around the earth's rotation axis may be written as (see Fig. 2.4)

$$\boldsymbol{\Omega} = \Omega(\mathbf{j}\cos\vartheta + \mathbf{k}\sin\vartheta). \tag{2.36}$$

If the scalar rotation rate is not Ω but is $u/(r\cos\vartheta)$, then the vector rotation rate is

$$\frac{u}{r\cos\vartheta}(\mathbf{j}\cos\vartheta + \mathbf{k}\sin\vartheta) = \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\vartheta}{r}. \tag{2.37}$$

Thus, the total rotation rate of a vector that moves with the flow is

$$\boldsymbol{\Omega}_{\text{flow}} = -\mathbf{i}\frac{v}{r} + \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\vartheta}{r}. \tag{2.38}$$

Applying (2.3) to (2.38), we find

$$\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} = \boldsymbol{\Omega}_{\text{flow}} \times \mathbf{i} = \frac{u}{r\cos\vartheta}(\mathbf{j}\sin\vartheta - \mathbf{k}\cos\vartheta), \tag{2.39a}$$

$$\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} = \boldsymbol{\Omega}_{\text{flow}} \times \mathbf{j} = -\mathbf{i}\frac{u}{r}\tan\vartheta - \mathbf{k}\frac{v}{r}, \tag{2.39b}$$

$$\frac{\mathrm{D}\mathbf{k}}{\mathrm{D}t} = \boldsymbol{\Omega}_{\text{flow}} \times \mathbf{k} = \mathbf{i}\frac{u}{r} + \mathbf{j}\frac{v}{r}. \tag{2.39c}$$

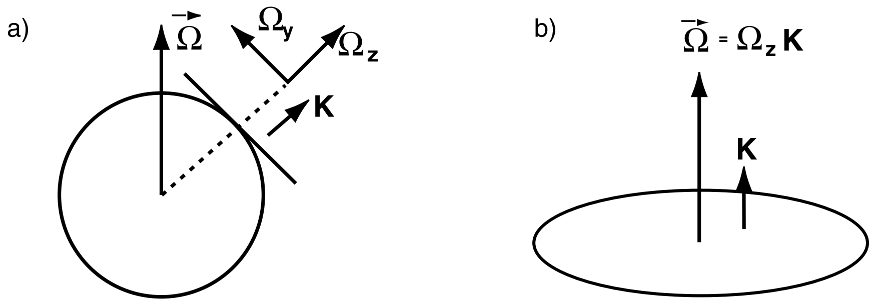

Fig. 2.4 a) On the sphere the rotation vector $\boldsymbol{\Omega}$ can be decomposed into two components, one in the local vertical and one in the local horizontal, pointing toward the pole. That is, $\boldsymbol{\Omega} = \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ where $\Omega_y = \Omega \cos\vartheta$ and $\Omega_z = \Omega \sin\vartheta$. In geophysical fluid dynamics, the rotation vector in the local vertical is often the more important component in the horizontal momentum equations. On a rotating disk, (b), the rotation vector $\boldsymbol{\Omega}$ is parallel to the local vertical $\mathbf{k}$.

2.2.3 Equations of motion

Mass Conservation and Thermodynamic Equation

The mass conservation equation, (1.38a), expanded in spherical co-odinates, is

$$\begin{aligned}\frac{\partial\rho}{\partial t} &+ \frac{u}{r\cos\vartheta}\frac{\partial\rho}{\partial\lambda} + \frac{v}{r}\frac{\partial\rho}{\partial\vartheta} + w\frac{\partial\rho}{\partial r} \\ &+ \frac{\rho}{r\cos\vartheta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{1}{r}\frac{\partial}{\partial r}(wr^2\cos\vartheta)\right] = 0\end{aligned} \tag{2.40}$$

Equivalently, using the form (1.38b), this is

$$\frac{\partial\rho}{\partial t} + \frac{1}{r\cos\vartheta}\frac{\partial(u\rho)}{\partial\lambda} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\rho\cos\vartheta) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w\rho) = 0. \tag{2.41}$$

The thermodynamic equation, (1.112), is a tracer advection equation. Thus, using (2.27), its (adiabatic) spherical coordinate form is

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\partial\theta}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial\theta}{\partial\lambda} + \frac{v}{r}\frac{\partial\theta}{\partial\vartheta} + w\frac{\partial\theta}{\partial r} = 0, \tag{2.42}$$

and similarly for tracers such as water vapour or salt.

Momentum Equation

Recall that inviscid momentum equation is:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi. \tag{2.43}$$

where Φ is the geopotential. In spherical coordinates the directions of the coordinate axes change with position and so the component expansion of (2.43) is

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\mathrm{D}u}{\mathrm{D}t}\mathbf{i} + \frac{\mathrm{D}v}{\mathrm{D}t}\mathbf{j} + \frac{\mathrm{D}w}{\mathrm{D}t}\mathbf{k} + u\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} + v\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} + w\frac{\mathrm{D}\mathbf{k}}{\mathrm{D}t} \tag{2.44a}$$

$$= \frac{Du}{Dt}\mathbf{i} + \frac{Dv}{Dt}\mathbf{j} + \frac{Dw}{Dt}\mathbf{k} + \boldsymbol{\Omega}_{\text{flow}} \times \boldsymbol{v} \tag{2.44b}$$

using (2.39). Using either (2.44a) and the expressions for the rates of change of the unit vectors given in (2.39), or (2.44b) and the expression for $\boldsymbol{\Omega}_{\text{flow}}$ given in (2.38), this becomes

$$\begin{aligned}\frac{D\boldsymbol{v}}{Dt} = \mathbf{i}\left(\frac{Du}{Dt} - \frac{uv\tan\vartheta}{r} + \frac{uw}{r}\right) + \mathbf{j}\left(\frac{Dv}{Dt} + \frac{u^2\tan\vartheta}{r} + \frac{vw}{r}\right)\\ + \mathbf{k}\left(\frac{Dw}{Dt} - \frac{u^2+v^2}{r}\right).\end{aligned} \tag{2.45}$$

Using the definition of a vector cross product the Coriolis term is:

$$\begin{aligned}2\boldsymbol{\Omega} \times \boldsymbol{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2\Omega\cos\vartheta & 2\Omega\sin\vartheta \\ u & v & w \end{vmatrix} \\ &= \mathbf{i}\,(2\Omega w\cos\vartheta - 2\Omega v\sin\vartheta) + \mathbf{j}\,2\Omega u\sin\vartheta - \mathbf{k}\,2\Omega u\cos\vartheta.\end{aligned} \tag{2.46}$$

Using (2.45) and (2.46), and the gradient operator given by (2.31), the momentum equation (2.43) becomes:

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\vartheta}\right)(v\sin\vartheta - w\cos\vartheta) = -\frac{1}{\rho r\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.47a}$$

$$\frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\vartheta}\right)u\sin\vartheta = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta}, \tag{2.47b}$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\Omega u\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g. \tag{2.47c}$$

The terms involving Ω are called Coriolis terms, and the quadratic terms on the left-hand sides involving $1/r$ are often called metric terms.

2.2.4 The primitive equations

The so-called *primitive equations* of motion are simplifications of the above equations frequently used in atmospheric and oceanic modelling.[3] Three related approximations are involved; these are:

(i) The hydrostatic approximation. In the vertical momentum equation the gravitational term is assumed to be balanced by the pressure gradient term, so that

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.48}$$

The advection of vertical velocity, the Coriolis terms, and the metric term $(u^2 + v^2)/r$ are all neglected.

(ii) The shallow-fluid approximation. We write $r = a + z$ where the constant a is the radius of the earth and z increases in the radial direction. The coordinate r is then replaced by a except where it used as the differentiating argument. Thus, for example,

$$\frac{1}{r^2}\frac{\partial(r^2 w)}{\partial r} \to \frac{\partial w}{\partial z}. \tag{2.49}$$

(iii) The traditional approximation. Coriolis terms in the horizontal momentum equations involving the vertical velocity, and the still smaller metric terms uw/r and vw/r, are neglected.

The second and third of these approximations should be taken, or not, together, the underlying reason being that they both relate to the presumed small aspect ratio of the motion, so the approximations succeed or fail together. If we make one approximation but not the other then we are being asymptotically inconsistent, and angular momentum and energy conservation are not assured [see section 2.2.7]. The hydrostatic approximation also depends on the small aspect ratio of the flow but in a slightly different way. For large-scale flow in the terrestrial atmosphere and ocean all three approximations are in fact all very accurate approximations. We defer a more complete treatment until section 2.7, in part because a treatment of the hydrostatic approximation is done most easily in the context of the Boussinesq equations, derived in section 2.4.

Making these approximations, the momentum equations become

$$\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega\sin\vartheta v - \frac{uv}{a}\tan\vartheta = -\frac{1}{a\rho\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.50a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + 2\Omega\sin\vartheta u + \frac{u^2\tan\vartheta}{a} = -\frac{1}{\rho a}\frac{\partial p}{\partial\vartheta}, \tag{2.50b}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{2.50c}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\vartheta} + w\frac{\partial}{\partial z}\right). \tag{2.51}$$

We note the ubiquity of the factor $2\Omega\sin\vartheta$, and take the opportunity to define the *Coriolis parameter*, $f \equiv 2\Omega\sin\vartheta$.

The corresponding mass conservation equation for a shallow fluid layer is:

$$\begin{aligned}&\frac{\partial\rho}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial\rho}{\partial\lambda} + \frac{v}{a}\frac{\partial\rho}{\partial\vartheta} + w\frac{\partial\rho}{\partial z}\\ &\qquad + \rho\left[\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{\partial w}{\partial z}\right] = 0,\end{aligned} \tag{2.52}$$

or equivalently,

$$\frac{\partial\rho}{\partial t} + \frac{1}{a\cos\vartheta}\frac{\partial(u\rho)}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\rho\cos\vartheta) + \frac{\partial(w\rho)}{\partial z} = 0. \tag{2.53}$$

2.2.5 Primitive equations in vector form

The primitive equations may be written in a compact vector form provided we make a slight reinterpretation of the material derivative of the coordinate axes. Let $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}$ be the horizontal velocity. The primitive equations (2.50a) and (2.50b) may be written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p \tag{2.54}$$

where $\boldsymbol{f} = f\mathbf{k} = 2\Omega \sin\vartheta \mathbf{k}$ and $\nabla_z p = [(a\cos\vartheta)^{-1}\partial p/\partial\lambda, a^{-1}\partial p/\partial\vartheta]$, the gradient operator at constant z. In (2.54) the material derivative of the horizontal velocity is given by

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mathbf{i}\frac{\mathrm{D}u}{\mathrm{D}t} + \mathbf{j}\frac{\mathrm{D}v}{\mathrm{D}t} + u\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} + v\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t}, \tag{2.55}$$

where instead of (2.39) we have

$$\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} = \widetilde{\boldsymbol{\Omega}}_{\text{flow}} \times \mathbf{i} = \mathbf{j}\frac{u\tan\vartheta}{a}, \tag{2.56a}$$

$$\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} = \widetilde{\boldsymbol{\Omega}}_{\text{flow}} \times \mathbf{j} = -\mathbf{i}\frac{u\tan\vartheta}{a}, \tag{2.56b}$$

where $\widetilde{\boldsymbol{\Omega}}_{\text{flow}} = \mathbf{k}u\tan\vartheta/a$ [which is the vertical component of (2.38), with r replaced by a.]. The advection of the horizontal wind $\boldsymbol{u}$ is still by the three-dimensional velocity $\boldsymbol{v}$. The vertical momentum equation is the hydrostatic equation, (2.50c), and the mass conservation equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{2.57}$$

where $\mathrm{D}/\mathrm{D}t$ on a scalar is given by (2.51), and the second expression is written out in full in (2.53).

2.2.6 The vector invariant form of the momentum equation

The 'vector invariant' form of the momentum equation is so-called because it appears to take the same form in all coordinate systems — there is no advective derivative of the coordinate system to worry about. Restricting attention to the incompressible case, with the aid of the identity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2)$ the three dimensional momentum equation may be written:

$$\frac{\partial\boldsymbol{v}}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega})\times\boldsymbol{v} = -\nabla B. \tag{2.58}$$

where $B = \phi + \boldsymbol{v}^2/2 + \Phi$ is the Bernoulli function and $\boldsymbol{\omega}$ is the relative vorticity, $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$. In spherical coordinates this is:

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{v} = \frac{1}{r^2\cos\vartheta}\begin{vmatrix} \mathbf{i}\,r\cos\vartheta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\vartheta & \partial/\partial r \\ ur\cos\vartheta & rv & w \end{vmatrix} \tag{2.59}$$

$$= \mathbf{i}\frac{1}{r}\left(\frac{\partial w}{\partial\vartheta} - \frac{\partial(rv)}{\partial r}\right) - \mathbf{j}\frac{1}{r\cos\vartheta}\left(\frac{\partial w}{\partial\lambda} - \frac{\partial}{\partial r}(ur\cos\vartheta)\right) + \mathbf{k}\frac{1}{r\cos\vartheta}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\vartheta}(u\cos\vartheta)\right). \tag{2.60}$$

Making the traditional approximation, and considering the horizontal and vertical components of the momentum equation separately, gives

$$\frac{\partial\boldsymbol{u}}{\partial t} + (2\boldsymbol{\Omega} + \mathbf{k}\zeta)\times\boldsymbol{u} + w\frac{\partial\boldsymbol{u}}{\partial z} = -\nabla_z B_h \tag{2.61}$$

where $\boldsymbol{u} = (u, v, 0)$, $\boldsymbol{\Omega} = \mathbf{k}\Omega \sin\vartheta$, $B_h = \phi + \boldsymbol{u}^2/2$, ∇_z is the horizontal gradient operator (the gradient at a constant value of z), and using (2.60), ζ is given by

$$\zeta = \frac{1}{a\cos\vartheta}\frac{\partial v}{\partial \lambda} - \frac{1}{a\cos\vartheta}\frac{\partial}{\partial \vartheta}(u\cos\vartheta) = \frac{1}{a\cos\vartheta}\frac{\partial v}{\partial \lambda} - \frac{1}{a}\frac{\partial u}{\partial \vartheta} + \frac{u}{a}\tan\vartheta. \tag{2.62}$$

The separate components of the momentum equation are then found to be:

$$\frac{\partial u}{\partial t} - (f + \zeta)v + w\frac{\partial u}{\partial z} = -\frac{1}{a\cos\vartheta}\frac{\partial B_h}{\partial \lambda}, \tag{2.63}$$

and

$$\frac{\partial v}{\partial t} + (f + \zeta)u + w\frac{\partial v}{\partial z} = -\frac{1}{a}\frac{\partial B_h}{\partial \vartheta}. \tag{2.64}$$

Similar expressions arise in a compressible fluid, with a different form for the right-hand side (problem 2.2).

2.2.7 Angular Momentum

The zonal momentum equation can be usefully expressed as a statement about axial angular momentum; that is, angular momentum about the rotation axis. The zonal angular momentum per unit mass is the component of angular momentum in the direction of the axis of rotation and it is given by, without making any shallow atmosphere approximation,

$$m = (u + \Omega r\cos\vartheta)r\cos\vartheta. \tag{2.65}$$

The evolution equation for this quantity follows from the zonal momentum equation and has the simple form

$$\frac{\mathrm{D}m}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} \tag{2.66}$$

where the material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \vartheta} + w\frac{\partial}{\partial r}. \tag{2.67}$$

Using the mass continuity equation, this can be written as

$$\frac{\mathrm{D}\rho m}{\mathrm{D}t} + \rho m\nabla\cdot\boldsymbol{v} = -\frac{\partial p}{\partial \lambda} \tag{2.68}$$

or

$$\frac{\partial \rho m}{\partial t} + \frac{1}{r\cos\vartheta}\frac{\partial(\rho u m)}{\partial \lambda} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial \vartheta}(\rho v m\cos\vartheta) + \frac{\partial}{\partial z}(\rho m w) = -\frac{\partial p}{\partial \lambda}. \tag{2.69}$$

This is an angular momentum conservation equation.

If the fluid is confined to a shallow layer near the surface of a sphere, then we may replace r, the radial coordinate, by a, the radius of the sphere, in the definition of m, and we define $\widetilde{m} \equiv (u + \Omega a\cos\vartheta)a\cos\vartheta$. Then (2.66) is replaced by

$$\frac{\mathrm{D}\widetilde{m}}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} \tag{2.70}$$

where now

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\vartheta} + w\frac{\partial}{\partial z}. \tag{2.71}$$

Using mass continuity this may be written as

$$\frac{\partial\rho\widetilde{m}}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial\widetilde{m}}{\partial\lambda} + \frac{v}{a}\frac{\partial\widetilde{m}}{\partial\vartheta} + w\frac{\partial\widetilde{m}}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial\lambda}. \tag{2.72}$$

which is the appropriate angular momentum conservation equation for a shallow atmosphere.

* *From angular momentum to the spherical component equations*

A somewhat indirect way to derive the three components of the momentum equation in spherical polar coordinates is to *begin* with (2.66) and the principle of conservation of energy. That is, we take the equations for conservation of angular momentum and energy as true *a priori* and demand that the forms of the momentum equation be constructed to satisfy these. Expanding the material derivative in (2.66), noting that $\mathrm{D}r/\mathrm{D}t = w$ and $\mathrm{D}\cos\vartheta/\mathrm{D}t = -(v/r)\sin\vartheta$, immediately gives (2.47a). Multiplication by u then yields

$$u\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega uv\sin\vartheta + 2\Omega uw\cos\vartheta - \frac{u^2v\tan\vartheta}{r} + \frac{u^2w}{r} = -\frac{u}{\rho r\cos\vartheta}\frac{\partial p}{\partial\lambda}. \tag{2.73}$$

Now suppose that the meridional and vertical momentum equations are of the form

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \text{Coriolis and metric terms} = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta} \tag{2.74a}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + \text{Coriolis and metric terms} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.74b}$$

but that we do not know what form the Coriolis and metric terms take. To determine that form, construct the kinetic energy equation by multiplying these equations by v and w. Now, the metric terms must vanish when we sum the resulting equations, so that (2.74a) must contain the Coriolis term $2\Omega u\sin\vartheta$ as well as the metric term $u^2\tan\vartheta/r$, and (2.74b) must contain a $-2\Omega u\cos\phi$ as well as the metric term u^2/r. But if (2.74b) contains the term u^2/r it must also contain the term v^2/r by isotropy, and therefore (2.74a) must also contain the term vw/r. In this way, (2.47) is precisely reproduced, although the skeptic might argue that the uniqueness of the form has not been proven.

A particular advantage of this approach arises in determining the appropriate momentum equations that conserve angular momentum and energy in the shallow-fluid approximation. We begin with (2.70) and expand to obtain (2.50a). Multiplying by u gives

$$u\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega uv\sin\vartheta - \frac{u^2v\tan\vartheta}{a} = -\frac{u}{\rho a\cos\vartheta}\frac{\partial p}{\partial\lambda}. \tag{2.75}$$

Evidently, the meridional momentum equation must contain the Coriolis term $2\Omega u\sin\vartheta$ and the metric term $u^2\tan\vartheta/a$, but the vertical momentum equation must have neither of the metric terms appearing in (2.47c). Thus we deduce the following equations:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega \sin\vartheta + \frac{u\tan\vartheta}{a}\right) v = -\frac{1}{\rho a\cos\vartheta}\frac{\partial p}{\partial \lambda} \tag{2.76a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \left(2\Omega \sin\vartheta + \frac{u\tan\vartheta}{a}\right) v = -\frac{1}{\rho a}\frac{\partial p}{\partial \vartheta} \tag{2.76b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g. \tag{2.76c}$$

This equation set, when used in conjunction with the thermodynamic and mass continuity equations, conserves appropriate forms of angular momentum and energy. In the hydrostatic approximation the material derivative of w in (2.76c) is *additionally* neglected. Thus, the hydrostatic approximation is mathematically and physically consistent with the shallow-fluid approximation, but it is an additional approximation with slightly different requirements that one may choose, rather than being required, to make. From an asymptotic perspective, the difference lies in the small parameter necessary for either approximation to hold, namely

$$\text{Shallow fluid and traditional approximations:} \qquad \gamma \equiv \frac{H}{a} \ll 1 \tag{2.77a}$$

$$\text{Small aspect ratio for hydrostatic approximation:} \qquad \alpha \equiv \frac{H}{L} \ll 1. \tag{2.77b}$$

where L is the horizontal scale of the motion and a is the radius of the earth. For hemispheric or global scale phenomena $L \sim a$ and the two approximations coincide. (Requirement (2.77b) for the hydrostatic approximation is derived in section 2.7.)

2.3 CARTESIAN APPROXIMATIONS: THE TANGENT PLANE

2.3.1 The f-plane

Although the rotation of the earth is central for many dynamical phenomena, the sphericity of the earth is not always so. This is especially true for phenomena on a scale somewhat smaller than global where the use of spherical coordinates becomes awkward, and it is more convenient to use a locally Cartesian representation of the equations. Referring to Fig. 2.4 we will define a plane tangent to the surface of the earth at a latitude ϑ_0, and then use a Cartesian coordinate system (x, y, z) to describe motion on that plane. For small excursions on the plane, $(x, y, z) \approx (a\lambda\cos\vartheta_0, a(\vartheta - \vartheta_0), z)$. Consistently, the velocity is $\boldsymbol{v} = (u, v, w)$, so that u,v and w are the components of the velocity *in the tangent plane.* These are approximately in the east-west, north-south and vertical directions, respectively.

The momentum equations for flow in this plane are then

$$\frac{\partial u}{\partial t} + (\boldsymbol{v}\cdot\nabla)u + 2\Omega_y w - 2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.78a}$$

$$\frac{\partial v}{\partial t} + (\boldsymbol{v}\cdot\nabla)v + 2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.78b}$$

$$\frac{\partial w}{\partial t} + (\boldsymbol{v}\cdot\nabla)w + 2(\Omega_x v - \Omega_y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.78c}$$

where the rotation vector $\boldsymbol{\Omega} = \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ and $\Omega_x = 0$, $\Omega_y = \Omega \cos\vartheta_0$ and $\Omega_z = \Omega \sin\vartheta_0$. If we make the traditional approximation, and so ignore the components of $\boldsymbol{\Omega}$ not in the direction of the local vertical, then

$$\frac{\mathrm{D}u}{\mathrm{D}t} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.79a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.79b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \rho g. \tag{2.79c}$$

where $f_0 = 2\Omega_z \sin\vartheta_0$. Defining the horizontal velocity vector $\boldsymbol{u} = (u, v, 0)$, the first two equations may also be written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}_0 \times \boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.80}$$

where $\mathrm{D}\boldsymbol{u}/\mathrm{D}t = \partial \boldsymbol{u}/\partial t + \boldsymbol{v} \cdot \nabla \boldsymbol{u}$, $\boldsymbol{f}_0 = 2\Omega \sin\vartheta_0 \mathbf{k} = f_0 \mathbf{k}$, and $\mathbf{k}$ is the direction perpendicular to the plane (it does not change its orientation with latitude). These equations are, evidently, exactly the same as the momentum equations in a system in which the rotation vector is aligned with the local vertical, as illustrated in the right panel in Fig. 2.4. They will describe flow on the surface of a rotating sphere to a good approximation provided the flow is of limited latitudinal extent so that the effects of sphericity are unimportant. This is known as the *f-plane* approximation since the Coriolis parameter is a constant. We may in addition make the hydrostatic approximation, in which case (2.79c) becomes the familiar $\partial p/\partial z = -\rho g$.

2.3.2 The beta-plane approximation

The magnitude of the vertical component of rotation varies with latitude, and this has important dynamical consequences. We can approximate this effect by allowing the effective rotation vector to vary. Thus, noting that, for small variations in latitude,

$$f = 2\Omega \sin\vartheta \approx 2\Omega \sin\vartheta_0 + 2\Omega \cos\vartheta_0 (\vartheta - \vartheta_0), \tag{2.81}$$

then on the tangent plane we may mimic this by allowing the Coriolis parameter to vary as

$$\boxed{f = f_0 + \beta y}, \tag{2.82}$$

where $f_0 = 2\Omega \sin\vartheta_0$ and $\beta = \partial f/\partial y = (2\Omega \cos\vartheta_0)/a$. This important approximation is known as the *beta-plane*, or *β-plane,* approximation. It captures the the most important *dynamical* effects of sphericity, without the complicating *geometric* effects, which are not essential to describe many phenomena. The momentum equations (2.79a), (2.79b) and (2.79c) (or its hydrostatic counterpart) are unaltered, save that f_0 is replaced by $f_0 + \beta y$ to represent a varying Coriolis parameter. Thus, sphericity combined with rotation is dynamically equivalent to a *differentially rotating* system. For future reference, we write down the β-plane horizontal momentum equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.83}$$

where $\boldsymbol{f} = (f_0 + \beta y)\hat{\mathbf{k}}$. In component form this equation becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.84a,b}$$

The mass conservation, thermodynamic and hydrostatic equations in the β-plane approximation are the same as the usual Cartesian (f-plane) forms of those equations.

2.4 EQUATIONS FOR A STRATIFIED OCEAN: THE BOUSSINESQ APPROXIMATION

The density variations in the ocean are quite small compared to the mean density, and we may exploit this to derive somewhat simpler but still quite accurate equations of motion. Let us first examine how much density does vary in the ocean.

2.4.1 Variation of density in the ocean

The variations of density in the ocean are due to three effects: the compression of water by pressure (which we denote as $\Delta_p\rho$), the thermal expansion of water if its temperature changes ($\Delta_T\rho$), and the haline contraction if its salinity changes ($\Delta_S\rho$). How big are these? An appropriate equation of state to approximately evaluate these effects is the linear one

$$\rho = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \frac{p}{\rho_0 c_s^2}\right], \tag{2.85}$$

where $\beta_T \approx 2\times10^{-4}\,\mathrm{K}^{-1}$, $\beta_S \approx 10^{-3}\,\mathrm{psu}^{-1}$ and $c_s \approx 1500\,\mathrm{m\,s^{-1}}$ (see the table on page 39). The three effects are then:

Pressure compressibility: We have $\Delta_p\rho \approx \Delta p/c^2 \approx \rho_0 gH/c^2$ where H is the depth and the pressure change is quite accurately evaluated using the hydrostatic approximation. Thus,

$$\frac{|\Delta_p\rho|}{\rho_0} \ll 1 \qquad \text{if} \qquad \frac{gH}{c^2} \ll 1, \tag{2.86}$$

or if $H \ll c^2/g$. The quantity $c^2/g \approx 200\,\mathrm{km}$ is the density scale height of the ocean. Thus, the pressure at the bottom of the ocean (say $H = 10\,\mathrm{km}$ in the deep trenches), enormous as it is, is insufficient to compress the water enough to make a significant change in its density. Changes in density due to dynamical variations of pressure are small if the Mach number is small, and this is also the case.

Thermal expansion: We have $\Delta_T\rho \approx -\beta_T\rho_0\Delta T$ and therefore

$$\frac{|\Delta_T\rho|}{\rho_0} \ll 1 \qquad \text{if} \qquad \beta_T\Delta T \ll 1. \tag{2.87}$$

For $\Delta T = 20\,\mathrm{K}$, $\beta_T\Delta T \approx 4\times10^{-3}$, and evidently we would require temperature differences of order β_T^{-1}, or 5000 K to obtain order one variations in density.

Saline contraction: We have $\Delta_S \rho \approx \beta_S \rho_0 \Delta S$ and therefore

$$\frac{|\Delta_S \rho|}{\rho_0} \ll 1 \qquad \text{if} \qquad \beta_S \Delta S \ll 1. \tag{2.88}$$

As changes is salinity in the ocean rarely exceed 5 psu, for which $\beta_S \Delta S = 5\times 10^{-3}$, the fractional change in the density of seawater is correspondingly very small.

Evidently, fractional density changes in the ocean are very small.

2.4.2 The Boussinesq equations

The *Boussinesq equations* are a set of equations that exploit the smallness of density variations in many liquids.[4] To set notation we write

$$\rho = \rho_0 + \delta\rho(x, y, z, t) \tag{2.89a}$$
$$= \rho_0 + \hat{\rho}(z) + \rho'(x, y, z, t) \tag{2.89b}$$
$$= \tilde{\rho}(z) + \rho'(x, y, z, t) \tag{2.89c}$$

where ρ_0 is a constant and we assume that

$$|\hat{\rho}|, |\rho'|, |\delta\rho| \ll \rho_0. \tag{2.90}$$

We need not assume that $|\rho'| \ll |\hat{\rho}|$, but this is often the case in the ocean. To obtain the Boussinesq equations we will just use (2.89a), but (2.89c) will be useful for the anelastic equations considered later.

Associated with the reference density is a reference pressure that is defined to be in hydrostatic balance with it. That is,

$$p = p_0(z) + \delta p(x, y, z, t) \tag{2.91a}$$
$$= \tilde{p}(z) + p'(x, y, z, t), \tag{2.91b}$$

where $|\delta p| \ll p_0$, $|p'| \ll \tilde{p}$ and

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} \equiv -g\rho_0, \qquad \frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} \equiv -g\tilde{\rho}. \tag{2.92a,b}$$

Note that $\nabla_z p = \nabla_z p' = \nabla_z \delta p$ and that $p_0 \approx \tilde{p}$ if $|\hat{\rho}| \ll \rho_0$.

Momentum equations

To obtain the Boussinesq equations we use $\rho = \rho_0 + \delta\rho$, and assume $\delta\rho/\rho_0$ is small. Without approximation, the momentum equation can be written as

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - \frac{\partial p_0}{\partial z}\mathbf{k} - g(\rho_0 + \delta\rho)\mathbf{k}, \tag{2.93}$$

and using (2.92a) this becomes, again without approximation,

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - g\delta\rho\mathbf{k}. \tag{2.94}$$

If density variations are small this becomes

$$\boxed{\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\Omega \times \boldsymbol{v}\right) = -\nabla\phi + b\mathbf{k}}, \tag{2.95}$$

where $\phi = \delta p/\rho_0$ and $b = -g\delta\rho/\rho_0$ is the *buoyancy*. Note that we should not and do not neglect the term $g\delta\rho$, for there is no reason to believe it to be small ($\delta\rho$ may be small, but g is big). Eq. (2.95) is the momentum equation in the Boussinesq approximation, and it is common to say that the Boussinesq approximation ignores all variations of density of a fluid in the momentum equation, except when associated with the gravitational term.

For most large-scale motions in the ocean the *deviation* pressure and density fields are also approximately in hydrostatic balance, and in that case the vertical component of (2.95) becomes

$$\frac{\partial\phi}{\partial z} = b. \tag{2.96}$$

A condition for (2.96) to hold is that vertical accelerations are small *compared to $g\delta\rho/\rho_0$, and not compared to the acceleration due to gravity itself.* For more discussion of this point, see section 2.7.

Mass Conservation

The unapproximated mass conservation equation is

$$\frac{\mathrm{D}\delta\rho}{\mathrm{D}t} + (\rho_0 + \delta\rho)\nabla\cdot\boldsymbol{v} = 0. \tag{2.97}$$

Provided that time scales advectively — that is to say that $\mathrm{D}/\mathrm{D}t$ scales in the same way as $\boldsymbol{v}\cdot\nabla$ — then we may approximate this equation by

$$\boxed{\nabla\cdot\boldsymbol{v} = 0}, \tag{2.98}$$

which is the same as that for a constant density fluid. This *absolutely does not* allow one to go back and use (2.97) to say that $D\delta\rho/Dt = 0$; the evolution of density is given by the thermodynamic equation in conjunction with an equation of state, and this should not be confused with the mass conservation equation. Note that in eliminating the time-derivative of density we eliminate the possibility of sound waves.

Thermodynamic equation and equation of state

The Boussinesq equations are closed by the addition of an equation of state, a thermodynamic equation and, as appropriate, a salinity equation. Neglecting salinity for the moment, a useful starting point is to write the thermodynamic equation, (1.120), as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} - \frac{1}{c^2}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\dot{Q}}{(\partial\eta/\partial\rho)_p T} \approx -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right) \tag{2.99}$$

using $(\partial\eta/\partial\rho)_p = (\partial\eta/\partial T)_p(\partial T/\partial\rho)_p \approx c_p/(T\rho_0\beta_T)$.

Given the expansions (2.89a) and (2.91a) this can be written as

$$\frac{D\delta\rho}{Dt} - \frac{1}{c^2}\frac{Dp_0}{Dt} = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right), \tag{2.100}$$

or, using (2.92a),

$$\frac{D}{Dt}\left(\delta\rho + \frac{\rho_0 g}{c^2}z\right) = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right), \tag{2.101}$$

as in (1.123). The severest approximation to this is to neglect the second term in brackets, and noting that $b = -g\delta\rho/\rho_0$ we obtain

$$\boxed{\frac{Db}{Dt} = \dot{b}}, \tag{2.102}$$

where $\dot{b} = g\beta_T\dot{Q}/c_p$. The momentum equation (2.95), mass continuity equation (2.98) and thermodynamic equation (2.102) then form a closed set, called the *simple Boussinesq equations*.

A somewhat more accurate approach is to include the compressibility of the fluid that is due to the hydrostatic pressure. Eq. (2.101) suggests that we define the potential density as $\delta\rho_{\text{pot}} = \delta\rho + \rho_0 gz/c_s^2$ the *potential buoyancy,* the buoyancy based on potential density, as

$$b_\sigma \equiv -g\frac{\delta\rho_{\text{pot}}}{\rho_0} = -\frac{g}{\rho_0}\left(\delta\rho + \frac{\rho_0 gz}{c_s^2}\right) = b - g\frac{z}{H_\rho}, \tag{2.103}$$

where $H_\rho = c_s^2/g$. The thermodynamic equation, (2.101), may be written

$$\frac{Db_\sigma}{Dt} = \dot{b}_\sigma, \tag{2.104}$$

where $\dot{b}_\sigma = \dot{b}$. Buoyancy itself is obtained from b_σ by the 'equation of state', $b = b_\sigma + gz/H_\rho$.

In many applications we may need to use a still more accurate equation of state. In that case (see section 1.5.5) we replace (2.102) by the thermodynamic equations

$$\boxed{\frac{D\theta}{Dt} = \dot{\theta}, \qquad \frac{DS}{Dt} = \dot{S}}, \tag{2.105a,b}$$

where θ is the potential temperature and S is salinity, along with an equation of state. This has the the general form $b = b(\theta, S, p)$, but to be consistent with the level of approximation in the other Boussinesq equations we should replace p by the hydrostatic pressure calculated with the reference density, that is by $-\rho gz$, and the equation of state takes the form

$$\boxed{b = b(\theta, S, z)}. \tag{2.106}$$

An example of (2.106) is (1.179) taken with the definition of buoyancy $b = -g\delta\rho/\rho_0$. The closed set of equations (2.95), (2.98), (2.105) and (2.106) are the *general Boussinesq equations.* (If we were to use the equation of state $b = b(\theta, S, p)$, we might call

Summary of Boussinesq Equations

The simple Boussinesq equations are, for an inviscid fluid:

Momentum equations: $$\frac{D\boldsymbol{v}}{Dt} + \boldsymbol{f} \times \boldsymbol{v} = -\nabla\phi + b\mathbf{k} \tag{B.1}$$

Mass conservation: $$\nabla \cdot \boldsymbol{v} = 0 \tag{B.2}$$

Buoyancy equation: $$\frac{Db}{Dt} = \dot{b} \tag{B.3}$$

A more general form replaces the buoyancy equation by:

Thermodynamic equation: $$\frac{D\theta}{Dt} = \dot{\theta} \tag{B.4}$$

Salinity equation: $$\frac{DS}{Dt} = \dot{S} \tag{B.5}$$

Equation of state: $$b = b(\theta, S, z) \tag{B.6}$$

the resulting equations the 'pseudo-Boussinesq' set.) Using an accurate equation of state and the Boussinesq approximation is the procedure used in many comprehensive ocean general circulation models. The Boussinesq equations, which with the hydrostatic and traditional approximations are often considered to be the oceanic primitive equations, are summarized in the shaded box.

* *Mean stratification and the buoyancy frequency*

The processes that cause density to vary in the vertical often differ from those that cause it to vary in the horizontal. For this reason it is sometimes useful to write $\rho = \rho_0 + \hat{\rho}(z) + \rho'(x, y, z, t)$ and define $\widetilde{b}(z) \equiv -g\hat{\rho}/\rho_0$ and $b' \equiv -g\rho'/\rho_0$. Using the hydrostatic equation to evaluate pressure, the thermodynamic equation (2.99) becomes, to a good approximation,

$$\frac{Db'}{Dt} + N^2 w = 0, \tag{2.107}$$

where

$$N^2(z) = \left(\frac{d\widetilde{b}}{dz} - \frac{g^2}{c_s^2}\right) = \frac{d\widetilde{b}_\sigma}{dz}. \tag{2.108}$$

where $\widetilde{b}_\sigma = \widetilde{b} - gz/H_\rho$. The quantity N^2 is a measure of the mean stratification of the fluid, and is equal to the vertical gradient of the mean potential buoyancy. N is known as the buoyancy frequency, something we return to in section 2.9. Equations (2.107) and (2.108) also hold in the simple Boussinesq equations, but with $c_s^2 = \infty$.

2.4.3 Energetics of the Boussinesq system

In a uniform gravitational field but with no other forcing or dissipation, we write the simple Boussinesq equations as

$$\frac{D\boldsymbol{v}}{Dt} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = b\mathbf{k} - \nabla\phi, \qquad \nabla\cdot\boldsymbol{v} = 0, \qquad \frac{Db}{Dt} = 0. \qquad (2.109a,b,c)$$

From (2.109a) and (2.109b) the kinetic energy density evolution (c.f., section 1.9) is given by

$$\frac{1}{2}\frac{Dv^2}{Dt} = bw - \nabla\cdot(\phi\boldsymbol{v}) \qquad (2.110)$$

where the constant reference density ρ_0 is omitted. Let us now define the potential Φ such that $\nabla\Phi = -\mathbf{k}$ (so $\Phi = -z$) and so

$$\frac{D\Phi}{Dt} = \nabla\cdot(\boldsymbol{v}\Phi) = -w. \qquad (2.111)$$

Using this and (2.109c) gives

$$\frac{D}{Dt}(b\Phi) = -wb. \qquad (2.112)$$

Adding this to (2.110) and expanding the material derivative gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi\right) + \nabla\cdot\left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi + \phi\right)\right] = 0. \qquad (2.113)$$

This constitutes an energy equation for the Boussinesq system, and may be compared to (1.194). (See also problem 2.12.) The energy density (divided by ρ_0) is just $\boldsymbol{v}^2/2 + b\Phi$. What does the second term represent? Its integral, multiplied by ρ_0, is the potential energy of the flow minus that of the basic state, or $\int g(\rho - \rho_0)z\,dz$. If there were a heating term on the right-hand side of (2.109c) this would directly provide a source of potential energy, rather than internal energy as in the compressible system. Because the fluid is incompressible, there is no conversion from kinetic and potential energy into internal energy.

** Energetics with a general equation of state*

Now consider the energetics of the general, adiabatic, Boussinesq equations. Suppose first that we allow the equation of state to be a function of pressure; the equations are motion are then (2.109) except that (2.109c) is replaced by

$$\frac{D\theta}{Dt} = 0, \qquad \frac{DS}{Dt} = 0, \qquad b = b(\theta, S, \phi). \qquad (2.114a,b,c)$$

A little algebraic experimentation will reveal that no energy conservation law of the form (2.113) generally exists for this system! The problem arises because, by requiring that the fluid be incompressible, we eliminate the proper conversion of internal energy to kinetic energy. However, if we use the consistent approximation $b = b(\theta, S, z)$, the system conserves an energy, as we now show.[5]

Define the potential, Π, by the integral of b at constant potential temperature and salinity

$$\Pi(\theta, S, z) \equiv -\int b\,dz. \qquad (2.115)$$

Taking its material derivative gives

$$\frac{\mathrm{D}\Pi}{\mathrm{D}t} = \left(\frac{\partial \Pi}{\partial \theta}\right)_{S,z} \frac{\mathrm{D}\theta}{\mathrm{D}t} + \left(\frac{\partial \Pi}{\partial S}\right)_{\theta,z} \frac{\mathrm{D}S}{\mathrm{D}t} + \left(\frac{\partial \Pi}{\partial z}\right)_{\theta,S} \frac{\mathrm{D}z}{\mathrm{D}t} = -bw, \tag{2.116}$$

using (2.114a,b). Combining (2.116) and (2.110) gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + \Pi\right) + \nabla \cdot \left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + \Pi + \phi\right)\right] = 0. \tag{2.117}$$

Thus, energetic consistency is maintained with an arbitrary equation of state, provided the pressure is replaced by a function of z. If b is not an explicit function of z in this equation of state, the conservation law is identical to (2.113).

2.5 EQUATIONS FOR A STRATIFIED ATMOSPHERE: THE ANELASTIC APPROXIMATION

2.5.1 Preliminaries

In the atmosphere the density varies significantly, especially in the vertical. However deviations of both ρ and p from a statically balanced state are often quite small, and the relative vertical variation of potential temperature is also small. We can usefully exploit these observations to give a somewhat simplified set of equations, useful both for theoretical and numerical analysis because sound waves are eliminated by way of an 'anelastic' approximation.[6] To begin we set

$$\rho = \widetilde{\rho}(z) + \delta\rho(x, y, z, t), \qquad p = \widetilde{p}(z) + \delta p(x, y, z, t) \tag{2.118}$$

where we assume that $|\delta\rho| \ll |\widetilde{\rho}|$ and we define $\widetilde{p}$ such that

$$\frac{\partial \widetilde{p}}{\partial z} \equiv -g\widetilde{\rho}(z). \tag{2.119}$$

The notation is similar to that for the Boussinesq case except that, importantly, the density basic state is now a (given) function of vertical coordinate. As with the Boussinesq case, the idea is to ignore dynamic variations of density (i.e., of $\delta\rho$) except where associated with gravity. First recall a couple of ideal gas relationships involving potential temperature, θ, and entropy s (divided by c_p, so $s \equiv \log\theta$), namely

$$s \equiv \log\theta = \log T - \frac{R}{c_p}\log p = \frac{1}{\gamma}\log p - \log\rho, \tag{2.120}$$

where $\gamma = c_p/c_v$, implying

$$\delta s = \frac{1}{\gamma}\frac{\delta p}{p} - \frac{\delta\rho}{\rho} \approx \frac{1}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta\rho}{\widetilde{\rho}} \tag{2.121}$$

Further, if $\widetilde{s} \equiv \gamma^{-1}\log\widetilde{p} - \log\widetilde{\rho}$ then

$$\frac{\mathrm{d}\widetilde{s}}{\mathrm{d}z} = \frac{1}{\gamma\widetilde{p}}\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}z} - \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}z} = -\frac{g\widetilde{\rho}}{\gamma\widetilde{p}} - \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}z}. \tag{2.122}$$

In the atmosphere, the left-hand side is, typically, much smaller than either of the two terms on the right-hand side.

2.5.2 The Momentum equation

The exact inviscid horizontal momentum equation is

$$(\widetilde{\rho} + \rho')\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \delta p. \tag{2.123}$$

Neglecting ρ' where it appears with $\widetilde{\rho}$ leads to

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi, \tag{2.124}$$

where $\phi = \delta p/\widetilde{\rho}$, and this is similar to the corresponding equation in the Boussinesq approximation.

The vertical component of the inviscid momentum equation is, without approximation,

$$(\widetilde{\rho} + \delta\rho)\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \widetilde{p}}{\partial z} - \frac{\partial \delta p}{\partial z} - g\widetilde{\rho} - g\delta\rho = -\frac{\partial \delta p}{\partial z} - g\delta\rho. \tag{2.125}$$

using (2.118). Neglecting $\delta\rho$ on the left-hand-side we obtain

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\widetilde{\rho}}\frac{\partial \delta p}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}} = -\frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial \widetilde{\rho}}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}}. \tag{2.126}$$

This is not a useful form for a gaseous atmosphere, since the variation of the mean density cannot be ignored. However, we may eliminate $\delta\rho$ in favour of δs using (2.121) to give

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{g}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial \widetilde{\rho}}{\partial z}, \tag{2.127}$$

and using (2.122) gives

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) + \frac{\mathrm{d}\widetilde{s}}{\mathrm{d}z}\frac{\delta p}{\widetilde{\rho}}. \tag{2.128}$$

What have these manipulations gained us? Two things:

(i) The gravitational term now involves δs rather than $\delta\rho$ which enables a more direct connection with the thermodynamic equation.

(ii) The potential temperature scale height ($\sim 100\,\mathrm{km}$) in the atmosphere is much larger than the density scale height ($\sim 10\,\mathrm{km}$), and so the last term in (2.128) is small.

The second item thus suggests that we choose our reference state to be one of constant potential temperature (see also problem 2.17). The term $\mathrm{d}\widetilde{s}/\mathrm{d}z$ then vanishes and the vertical momentum equation becomes

$$\boxed{\frac{\mathrm{D}w}{\mathrm{D}t} = g\delta s - \frac{\partial \phi}{\partial z}}, \tag{2.129}$$

where $\phi = \delta p/\widetilde{\rho}$, $\delta s = \delta\theta/\widetilde{\theta}$ and $\widetilde{\theta} = \theta_0$, a constant. If we define a buoyancy by $b_a \equiv g\delta s = g\delta\theta/\widetilde{\theta}$, then (2.124) and (2.129) have the same form as the Boussinesq momentum equations, but with different definitions of b and ϕ.

2.5.3 Mass conservation

Using (2.118a) the mass conservation equation may be written, without approximation, as

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot [(\widetilde{\rho} + \delta\rho)\boldsymbol{v}] = 0. \tag{2.130}$$

We neglect $\delta\rho$ where it appears with $\widetilde{\rho}$ in the divergence term. Further, the local time derivative will be small if time itself is scaled advectively (i.e., $T \sim L/U$ and sound waves do not dominate), giving

$$\nabla \cdot \boldsymbol{u} + \frac{1}{\widetilde{\rho}}\frac{\partial}{\partial z}(\widetilde{\rho} w) = 0 \tag{2.131}$$

It is here that the eponymous 'anelastic approximation' arises: the elastic compressibility of the fluid is neglected, and this serves to eliminate sound waves. For reference, in spherical coordinates the equation is

$$\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial \lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial \vartheta}(v\cos\vartheta) + \frac{1}{\widetilde{\rho}}\frac{\partial(w\widetilde{\rho})}{\partial z} = 0. \tag{2.132}$$

In an ideal gas, the choice of constant potential temperature determines how the reference density $\widetilde{\rho}$ varies with height. In some circumstances it is convenient to let $\widetilde{\rho}$ be a constant, ρ_0 (effectively choosing a different equation of state), in which case the anelastic equations become identical with the Boussinesq equations, although we may continue to interpret the buoyancy in terms of potential temperature.

2.5.4 Thermodynamic equation

The thermodynamic equation for an ideal gas may be written

$$\frac{\mathrm{D}\ln\theta}{\mathrm{D}t} = \frac{\dot{Q}}{Tc_p}. \tag{2.133}$$

In the anelastic equations, $\theta = \widetilde{\theta} + \delta\theta$ where $\widetilde{\theta}$ is constant, and the thermodynamic equation is

$$\frac{\mathrm{D}\delta s}{\mathrm{D}t} = \frac{\widetilde{\theta}}{Tc_p}\dot{Q}. \tag{2.134}$$

Summarizing, the complete set of anelastic equations, with rotation but with no dissipation or diabatic terms, is

$$\boxed{\begin{gathered}\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = \mathbf{k}b_a - \nabla\phi \\ \frac{\mathrm{D}b_a}{\mathrm{D}t} = 0 \\ \nabla\cdot(\widetilde{\rho}\boldsymbol{v}) = 0\end{gathered}}\,, \tag{2.135}$$

where $b_a = g\delta s = g\delta\theta/\widetilde{\theta}$. The main difference between the anelastic and Boussinesq sets of equations is in the mass continuity equation, and when $\widetilde{\rho} = \rho_0 =$

constant the two equation sets are formally identical. However, whereas the Boussinesq approximation is a very good one for ocean dynamics, the anelastic approximation is much less so for large-scale atmosphere flow: the constancy of the reference potential temperature state is then not a particularly good approximation and so the deviations in density from its reference profile are not especially small, leading to inaccuracies in the momentum equation. Nevertheless, the anelastic equations have been used very productively in limited area 'large-eddy-simulations' where one does not wish to make the hydrostatic approximation but where sound waves are unimportant.[7] The equations also provide a good jumping-off point for theoretical studies and the still simpler models that will be considered in the chapter 5.

2.5.5 * Energetics of the anelastic equations

Conservation of energy follows in much the same way as for the Boussinesq equations, except that $\widetilde{\rho}$ enters. Take the dot product of (2.135a) with $\widetilde{\rho}\boldsymbol{v}$ to obtain

$$\widetilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2\right) = -\nabla\cdot(\phi\widetilde{\rho}\boldsymbol{v}) + b_a\widetilde{\rho}w \tag{2.136}$$

Now, define a potential $\Phi(z)$ such that $\nabla\Phi = -\mathbf{k}$, and so

$$\widetilde{\rho}\frac{\mathrm{D}\Phi}{\mathrm{D}t} = -w\widetilde{\rho}. \tag{2.137}$$

Combining this with the thermodynamic equation (2.135b) gives

$$\widetilde{\rho}\frac{\mathrm{D}(b_a\Phi)}{\mathrm{D}t} = -wb_a\widetilde{\rho}. \tag{2.138}$$

Adding this to (2.136) gives

$$\widetilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right) = -\nabla\cdot(\phi\widetilde{\rho}\boldsymbol{v}), \tag{2.139}$$

or, expanding the material derivative,

$$\frac{\partial}{\partial t}\left[\widetilde{\rho}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right)\right] + \nabla\cdot\left[\widetilde{\rho}\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi + \phi\right)\right] = 0. \tag{2.140}$$

This equation has the form

$$\frac{\partial E}{\partial t} + \nabla\cdot[\boldsymbol{v}(E + \widetilde{\rho}\phi)] = 0 \tag{2.141}$$

where $E = \widetilde{\rho}(\boldsymbol{v}^2/2 + b_a\Phi)$ is the energy density of the flow. This is a consistent energetic equation for the system, and when integrated over a closed domain the total energy is evidently conserved. The total energy density comprises the kinetic energy and a term $\widetilde{\rho}b_a\Phi$, which is analogous to the potential energy of a Boussinesq system. However, it is not exactly equal to that because b_a is the bouyancy based on potential temperature, not density; rather, the term combines contributions from both the internal energy and the potential energy.

2.6 CHANGING VERTICAL COORDINATE

Although using z as a vertical coordinate is a natural choice given our Cartesian worldview, it is not the only option, nor is it always the most useful one. Any variable that has a one-to-one correspondence with z in the vertical, so any variable that varies monotonically with z, could be used; pressure and, perhaps surprisingly, entropy, are common choices. In the atmosphere pressure almost always falls monotonically with height, and using it instead of z provides a useful simplification of the mass conservation and geostrophic relations, as well as a more direct connection with observations, which are often taken at fixed values of pressure. (In the ocean pressure is almost the same as height, because density is almost constant.) Entropy seems an exotic vertical coordinate, but it is very useful in adiabatic flow, and we consider that in chapter 3.

2.6.1 Pressure coordinates

The primitive equations of motion for an ideal gas can be written,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p, \tag{2.142a}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2.142b}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{2.142c}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0, \tag{2.142d}$$

where $p = \rho RT$ and $\theta = T\,(p_{\mathrm{R}}/p)^{R/c_p}$, and p_{R} is the reference pressure. These equations can be put into a form similar to the Boussinesq equations by transforming from Cartesian [i.e., (x, y, z)] to *pressure coordinates*, (x, y, p). The analog of the vertical velocity is $\omega \equiv Dp/Dt$, and the advective derivative itself is given by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla_p + \omega\frac{\partial}{\partial p}. \tag{2.143}$$

The horizontal and time derivatives are taken at constant pressure. However, x and y are still purely horizontal coordinates, and $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$ is still a strictly horizontal velocity, perpendicular to the vertical (z) axis. The operator $\mathrm{D}/\mathrm{D}t$ is of course the same in pressure or height coordinates because is simply the total derivative of some property of a fluid parcel. However, the individual terms comprising it in general differ between height and pressure coordinates.

To obtain an expression for the pressure force, first consider a general vertical coordinate, ξ. Then the chain rule gives

$$\left(\frac{\partial}{\partial x}\right)_\xi = \left(\frac{\partial}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_\xi \frac{\partial}{\partial z}. \tag{2.144}$$

Now let $\xi = p$ and apply the relationship to p itself to give

$$0 = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z}, \tag{2.145}$$

which, using the hydrostatic relationship, gives

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho \left(\frac{\partial \Phi}{\partial x}\right)_p, \tag{2.146}$$

where $\Phi = gz$ is the *geopotential*. Thus, the horizontal pressure force in the momentum equations is

$$\frac{1}{\rho}\nabla_z p = \nabla_p \Phi, \tag{2.147}$$

where the subscripts on the gradient operator indicate that the horizontal derivatives are taken at constant z or constant p. Also, from (2.142b), the hydrostatic equation is just

$$\frac{\partial \Phi}{\partial p} = -\alpha. \tag{2.148}$$

Mass continuity

The mass conservation equation simplifies attractively in pressure coordinates, if the hydrostatic approximation is used. Recall that the mass conservation equation can be derived from the Lagrangian form

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho\delta V = 0, \tag{2.149}$$

where $\delta V = \delta x \delta y \delta z$ is a volume element. But by the hydrostatic relationship $\rho \delta z = (1/g)\delta p$ and thus

$$\frac{\mathrm{D}}{\mathrm{D}t}(\delta x \delta y \delta p) = 0. \tag{2.150}$$

This is completely analogous to the expression for the Lagrangian conservation of volume in an incompressible fluid, (1.15). Thus, without further ado, we write the mass conservation in pressure coordinates as

$$\nabla_p \cdot \boldsymbol{u} + \frac{\partial \omega}{\partial p} = 0, \tag{2.151}$$

where the horizontal derivative is taken at constant pressure. (See also problem 2.20.) The primitive equations in pressure coordinates are thus:

$$\boxed{\begin{aligned} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} &= -\nabla_p \Phi \\ \frac{\partial \Phi}{\partial p} &= -\alpha \\ \frac{\mathrm{D}\theta}{\mathrm{D}t} &= 0 \\ \nabla_p \cdot \boldsymbol{u} + \frac{\partial \omega}{\partial p} &= 0 \end{aligned}}, \tag{2.152}$$

where $\mathrm{D}/\mathrm{D}t$ is given by (2.143). The equation set is completed with the addition of the ideal gas equation and the definition of potential temperature. These are not quite isomorphic to the Boussinesq equations, because the hydrostatic equation is $\partial\Phi/\partial p = -\alpha = -(\theta R/p_{\mathrm{R}})(p_{\mathrm{R}}/p)^{1/\gamma}$ and not, as we would require, $\partial\Phi/\partial p = -\theta$.

The main practical difficulty with these equations is the lower boundary condition. Using

$$w \equiv \frac{Dz}{Dt} = \frac{\partial z}{\partial t} + \boldsymbol{u} \cdot \nabla_p z + \omega \frac{\partial z}{\partial p}, \tag{2.153}$$

and (2.148), the boundary condition of $w = 0$ at $z = z_s$ becomes

$$\frac{\partial \Phi}{\partial t} + \boldsymbol{u} \cdot \nabla_p \Phi - \alpha \omega = 0 \tag{2.154}$$

at $p(x, y, z_s, t)$. In theoretical studies, it is common to assume that the lower boundary is in fact a constant pressure surface and simply assume that $\omega = 0$, or sometimes the condition $\omega = -\alpha^{-1} \partial \Phi / \partial t$ is used. For realistic studies (with general circulation models, say) the fact that the level $z = 0$ is not a coordinate surface must be properly accounted for. For this reason, and especially if the lower boundary is uneven because of the presence of topography, so-called *sigma coordinates* are sometimes used, in which the vertical coordinate is chosen so that the lower boundary is a coordinate surface. Sigma coordinates may use height itself as a measure of displacement (typical in oceanic applications) or use pressure (typical in atmospheric applications). In the latter case the vertical coordinate is $\sigma = p/p_s$ where $p_s(x, y, t)$ is the surface pressure. The difficulty of applying (2.154) is replaced by a prognostic equation for the surface pressure, which is derived from the mass conservation equation (problem 2.21).

Log-pressure coordinates

A variant of pressure coordinates is *log-pressure* coordinates, in which the vertical coordinate is $Z = -H \ln(p/p_R)$ where p_R is a reference pressure (say 1000 mb) and H a constant (for example the scale height RT_s/g) so that Z has units of length. The 'vertical velocity' for the system is now

$$W \equiv \frac{DZ}{Dt} \tag{2.155}$$

and the advective derivative is now

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla_p + W \frac{\partial}{\partial Z}, \tag{2.156}$$

(Capital letters are conventionally used for some variables in log-pressure coordinates, and these are not to be confused with scaling parameters.) It is straightforward to show (problem 2.22) that the primitive equations of motion in these coordinates are:

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_Z \Phi \tag{2.157a}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{RT}{H} \tag{2.157b}$$

$$\frac{D\theta}{Dt} = 0 \tag{2.157c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial Z} - \frac{W}{H} = 0. \tag{2.157d}$$

The last equation may be written $\nabla_Z \cdot \boldsymbol{u} + \rho_*^{-1} \partial(\rho_* W)/\partial z = 0$, where $\rho_* = \exp(-z/H)$, a form similar to the mass conservation equation in the anelastic equations.

2.7 HYDROSTATIC BALANCE

In this section and the next we consider two of the most fundamental balances in geophysical fluid dynamics — hydrostatic balance and geostrophic balance. Neither are exact, but their approximate satisfaction has profound consequences on the behaviour of the atmosphere and ocean.

2.7.1 Preliminaries

Consider the relative sizes of terms in (2.78c),

$$\frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} + \Omega U \sim \frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{2.158}$$

For most large-scale motion in the atmosphere and ocean the terms on the right-hand side are orders of magnitude larger than those on the left, and therefore must be approximately equal. Explicitly, suppose $W \sim 1\,\mathrm{cm\,s^{-1}}$, $L \sim 10^5\,\mathrm{m}$, $H \sim 10^3\,\mathrm{m}$, $U \sim 10\,\mathrm{m\,s^{-1}}$, $T = L/U$. Then by substituting into (2.158) it seems that the pressure term is the only one which could balance the gravitational term, and we are led to the following vertical momentum equation,

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2.159}$$

otherwise known as *hydrostatic balance*.

However, (2.159) is not necessarily a useful equation! Let us suppose that the density is a constant, ρ_0 ; we can then write the pressure as

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t), \tag{2.160}$$

where

$$\frac{\partial p_0}{\partial z} \equiv -\rho_0 g. \tag{2.161}$$

That is, p_0 and ρ_0 are in hydrostatic balance. The vertical momentum equation becomes, without approximation,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \nu\nabla^2 w. \tag{2.162}$$

Thus, for constant density fluids, the gravitational term has no dynamical effect: there is no buoyancy force, and the pressure term in the horizontal momentum equations can be replaced by p'. Hydrostatic balance, and in particular (2.161), is certainly not an appropriate vertical momentum equation in this case. If the fluid is stratified, we should therefore subtract off the hydrostatic pressure associated with the mean density before we can determine whether hydrostasy is a useful *dynamical* approximation, accurate enough to determine the horizontal pressure gradients. This is automatic in the Boussinesq equations, where the vertical momentum equation is

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b. \tag{2.163}$$

and the hydrostatic balance of the basic state is already subtracted out. In the more general equation,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.164}$$

we need to compare the advective term on the left-hand side with the pressure variations arising from horizontal flow in order to determine whether hydrostasy is an appropriate vertical momentum equation. Of course, if we simply need to determine the pressure for use in an equation of state then we simply need to compare the sizes of the dynamical terms in (2.78c) with g itself in order to determine whether a hydrostatic approximation will suffice.

2.7.2 Scaling and the aspect ratio

In a non-rotating Boussinesq fluid the horizontal momentum equation $\mathrm{D}\boldsymbol{u}/\mathrm{D}t = -\nabla\phi$ implies the scaling

$$\phi \sim U^2 \tag{2.165}$$

If we now use mass conservation to scale vertical velocity, so that

$$\frac{W}{H} \sim \frac{U}{L} \tag{2.166}$$

then the advective terms in the vertical momentum equation all scale as

$$\frac{\mathrm{D}w}{\mathrm{D}t} \sim \frac{UW}{L} = \frac{U^2 H}{L^2} \tag{2.167}$$

Using (2.165) and (2.167) the ratio of the advective term to the pressure gradient term in the vertical momentum equations then scales as

$$\frac{|Dw/Dt|}{|(1/\rho)\partial\phi/\partial z|} \sim \frac{U^2 H/L^2}{U^2/H} \sim \left(\frac{H}{L}\right)^2. \tag{2.168}$$

Thus, the condition for hydrostasy is:

$$\boxed{\alpha^2 \equiv \left(\frac{H}{L}\right)^2 \ll 1}, \tag{2.169}$$

in which case the advective term in the vertical momentum is small. Thus, hydrostasy is an *aspect ratio approximation;* it holds when the aspect ratio $\alpha \equiv H/L$ is small.

Effects of rotation

In the presence of rapid rotation geostrophic-balance suggests the pressure scaling $\phi \sim fUL$ holds and we obtain

$$\frac{|Dw/Dt|}{|(1/\rho)\partial p/\partial z|} \sim \frac{WH}{fL^2} = Ro\frac{WH}{UL}. \tag{2.170}$$

Furthermore, in geostrophic balance the horizontal flow may be nearl non-divergent (as we see in the next section), so that $W \sim UH/L$ is an overestimate of the magnitude of the vertical velocity. Let us thus write $W \sim \epsilon UH/L$ where $\epsilon \ll 1$. (Later on we'll see that ϵ is in fact related to the Rossby number.) Using this in (2.170) gives

$$\frac{|Dw/Dt|}{|(1/\rho)\partial p/\partial z|} \sim \epsilon Ro \left(\frac{H}{L}\right)^2, \tag{2.171}$$

which is evidently very small for the large-scale flow. In particular, because $\epsilon Ro \ll 1$, rotation tends to weaken further the conditions needed for hydrostasy; that is, a rapidly rotating fluid is more likely to be in hydrostatic balance than a non-rotating fluid, other conditions being equal.

** Effects of stratification*

The above results say little about the dynamics that might give rise to hydrostatic or non-hydrostatic flow. Our intuition suggests that hydrostatic balance might be questionable in small-scale convective activity where the vertical velocity is high, and most applicable in highly stratified flow for then vertical velocity is diminished. But in highly stratified flow the estimate of W from (2.167) may be too restrictive. Furthermore, the vertical scale H is not always known *a priori,* for it need not be the domain scale. In a laboaratory rotating tank, for example, the aspect ratio of the fluid is $\mathcal{O}(1)$, but the vertical scale of the motion is much smaller. We thus give another estimate of the vertical velocity that takes explicit account of known stratification and use it to derive a slightly weaker condition for hydrostasy.

To obtain an estimate of the vertical velocity, we use the Boussinesq approximation with a vertical momentum equation:

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b. \tag{2.172}$$

For hydrostatic balance we demand

$$\frac{UW}{L} \ll \Delta b, \tag{2.173}$$

where Δb is the scaling magnitude for b. It is only the horizontal variations of b that matter, so let us assume that the magnitude for w is given by the thermodynamic equation written in the form

$$\frac{\mathrm{D}b'}{\mathrm{D}t} + N^2 w = 0, \tag{2.174}$$

where the mean stratification (N^2) is given — determined, for example, by the larger scale circulation. Then,

$$W \sim \frac{U\Delta b}{LN^2} \tag{2.175}$$

This implies that $W/H = (U/L)\,\Delta b/(HN^2)$, and, assuming that $\Delta b/H \ll N^2$, that $W/H \ll U/L$ and horizontal advection dominates vertical advection of the buoyancy anomaly. Using (2.175), the condition for hydrostatic balance, (2.173), becomes

$$\boxed{\frac{U^2}{L^2N^2} \ll 1}. \tag{2.176}$$

Since the buoyancy frequency N is a measure of stratification (the higher the frequency, the more stratified the fluid), (2.176) formalizes our intuitive expectation that the more stratified a fluid the more vertical motion is suppressed and the more likely hydrostatic balance is to hold.

The *Froude number* may be defined by

$$F \equiv \frac{U}{NH}. \tag{2.177}$$

Then

$$\frac{U^2}{L^2 N^2} = F^2 \frac{H^2}{L^2}, \tag{2.178}$$

and using (2.176) the hydrostatic condition is

$$\boxed{F^2 \alpha^2 \ll 1}, \tag{2.179}$$

where $\alpha = H/L$ is the aspect ratio of the motion. Thus, for a given Froude number a small aspect ratio will favor hydrostasy. This derivation differs from that leading to (2.169) in its use of the thermodynamic equation, rather than the mass conservation equation, to give an estimate of the vertical velocity.

Why bother with any of this scaling? Why not just say that hydrostatic balance holds when $|Dw/Dt| \ll |b|$? One reason is that don't really have a good idea of the value of W from direct measurements, and it may change significantly in different oceanic and atmospheric parameter regimes. On the other hand the Froude number and the aspect ratio are familiar nondimensional parameters with a wide applicability in other contexts, and which we can control in a laboratory setting or estimate more easily in the ocean or atmosphere. Still, as in most scaling theory, deciding which parameters are given and which should be derived is often a *choice*, rather than being set *a priori*.

Hydrostatic balance in waves

If the motion is predominantly wavelike, rather than advective, then the advective derivative scales like a frequency: $D/Dt \sim \omega$. Using the vertical momentum equation we then require

$$\omega w \ll b, \tag{2.180}$$

and the thermodynamic equation has the scaling

$$\omega b \sim N^2 w. \tag{2.181}$$

These together demand that

$$\omega^2 \ll N^2 \tag{2.182}$$

for hydrostatic balance to hold. Buoyancy oscillations have $\omega \approx N$ and are essentially non-hydrostatic.

Oceanic applicability

For the large scale ocean circulation, let $N \sim 10^{-2}\,\mathrm{s}^{-1}$, $U \sim 0.1\,\mathrm{m\,s}^{-1}$ and $H \sim 1\,\mathrm{km}$. Then

$$F = \frac{U}{NH} \sim 10^{-2} \ll 1 \tag{2.183}$$

Thus, $F^2\gamma^2 \ll 1$ even for unit aspect-ratio motion. For gyre scale flow $L \sim 10^6\,\mathrm{m}$ and $F^2\gamma^2 \sim 10^{-10}$ and hydrostatic balance is a very good approximation indeed.

For intense convection, for example in the Labrador Sea, the hydrostatic approximation may be less appropriate. The intense descending plumes may have an aspect ratio (H/L) of one or greater and the stratification is very weak: the hydrostatic condition is then simply a requirement that stratification is sufficiently strong that the Froude number is small. Representative orders of magnitude are $U \sim W \sim 0.1\,\mathrm{m\,s}^{-1}$, $H \sim 1\,\mathrm{km}$ and $N \sim 10^{-3}\,\mathrm{s}^{-1}$–$10^{-4}\,\mathrm{s}^{-1}$. For these values F ranges between 0.1 and 1, and at the upper end of this range hydrostatic balance is violated.

Atmospheric applicability

Similar considerations apply to the atmosphere, although the details differ. Over much of the lower atmosphere $N \sim 10^{-2}\,\mathrm{s}^{-1}$ so that with $U = 10\,\mathrm{m\,s}^{-1}$ and $H = 1\,\mathrm{km}$ for large-scale flow $F \sim 1$. Hydrostasy is then maintained because the aspect ratio H/L is much less than unity. For smaller scale atmospheric motion associated with fronts and, especially, convection, there can be little expectation that hydrostatic balance will be a good approximation.

2.8 GEOSTROPHIC AND THERMAL WIND BALANCE

We now consider the dominant dynamical balance in the horizontal components of the momentum equation. In the horizontal plane (meaning along geopotential surfaces) we find that the Coriolis term is much larger than the advective terms and the dominant balance is between it and the horizontal pressure force. This balance is called *geostrophic balance,* and it occurs when the Rossby number is small, as we now investigate.

2.8.1 The Rossby Number

The *Rossby number* characterizes the importance of rotation in a fluid.[8] It is, essentially, the ratio of the magnitude of the relative acceleration to the Coriolis acceleration, and it is of fundamental importance in geophysical fluid dynamics. It arises from a simple scaling of horizontal momentum equation, namely

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.184a}$$

$$U^2/L \qquad fU \tag{2.184b}$$

where U is the approximate magnitude of the horizontal velocity and L is a typical lengthscale over which that velocity varies. (We assume that $W/H \lesssim U/L$, so that

Variable	Scaling Symbol	Meaning	Atmos. value	Ocean value
(x, y)	L	Horizontal length	10^6 m	10^5 m
t	T	Timescale	1 day (10^5 s)	10 days (10^6 s)
(u, v)	U	Horizontal velocity	$10\,\mathrm{m\,s^{-1}}$	$0.1\,\mathrm{m\,s^{-1}}$
	Ro	Rossby number, U/fL	0.1	0.01

Table 2.1 Scales of large-scale flow in atmosphere and ocean. The choices given are representative of large-scale eddying motion in both systems.

vertical advection does not dominate the advection.) The ratio of the sizes of the advective and Coriolis terms is defined to be the Rossby number,

$$\boxed{Ro \equiv \frac{U}{fL}}. \tag{2.185}$$

If the Rossby number is small then rotation effects are important, and as the values in table 2.1 indicate this is the case for large-scale flow in both ocean and atmosphere.

Another intuitive way to think about the Rossby number is in terms of timescales. The Rossby number based on a timescale is

$$Ro_t \equiv \frac{1}{fT} \tag{2.186}$$

where T is a timescale associated with the dynamics at hand. If the timescale is an advective one, meaning that $T \sim L/U$, then this definition is equivalent to (2.185). Now, $f = 2\Omega \sin\vartheta$, where Ω is the angular velocity of the rotating frame and equal to $2\pi \sin\vartheta/T_p$ where T_p is the period of rotation (24 hours). Thus,

$$Ro_t = \frac{T_p}{4\pi T \sin\vartheta} = \frac{T_i}{T} \tag{2.187}$$

where $T_i = 1/f$ is the 'inertial timescale', about three hours in midlatitudes. Thus, for phenomena with timescales much longer than this, such as the motion of the Gulf Stream or a mid-latitude atmospheric weather system, the effects of the earth's rotation can be expected to be important, whereas a short-lived phenomena, such as a cumulus cloud or tornado, may be oblivious to such rotation. The expressions (2.185) and (2.186) of course, just approximate measures of the importance of rotation.

2.8.2 Geostrophic Balance

If the Rossby number is sufficiently small in (2.184a) then the rotation term will dominate the nonlinear advection term, and if the time period of the motion scales

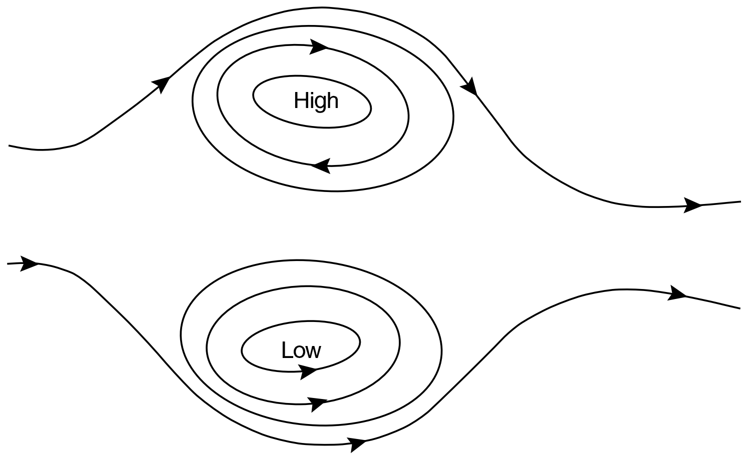

Fig. 2.5 Schematic of geostrophic flow with a positive value of the Coriolis parameter f. Flow is parallel to the lines of constant pressure (isobars). Cyclonic flow is anticlockwise around a low pressure region and anticyclonic flow is clockwise around a high. If f were negative, as in the Southern hemisphere, (anti-)cyclonic flow would be (anti-)clockwise.

advectively then the rotation term also dominates the local time derivative. The only term which can then balance the rotation termis the pressure term, and therefore we must have

$$\boldsymbol{f} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla_z p, \tag{2.188}$$

or, in Cartesian component form

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2.189}$$

This balance is known as *geostrophic balance,* and its consequences are profound, giving geophysical fluid dynamics a special place in the broader field of fluid dynamics. We *define* the geostrophic velocity by

$$\boxed{fu_g \equiv -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv_g \equiv \frac{1}{\rho}\frac{\partial p}{\partial x}}, \tag{2.190}$$

and for low Rossby number flow $u \approx u_g$ and $v \approx v_g$. In spherical coordinates the geostrophic velocity is

$$fu_g = -\frac{1}{\rho a}\frac{\partial p}{\partial \vartheta}, \qquad fv_g = \frac{1}{a\rho\cos\vartheta}\frac{\partial p}{\partial \lambda}, \tag{2.191}$$

where $f = 2\Omega\sin\vartheta$. Geostrophic balance has a number of immmediate ramifications:

- ⋆ Geostrophic flow is parallel to lines of constant pressure (isobars). If $f > 0$ the flow is anti-clockwise round a region of low pressure and clokwise around a region of high pressure (see Fig. 2.5).

⋆ If the Coriolis force is constant and if the density does not vary in the horizontal the geostrophic flow is horizontally non-divergent and

$$\boxed{\nabla_z \cdot \boldsymbol{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0}. \tag{2.192}$$

We may define the *streamfunction*, ψ, by

$$\psi \equiv \frac{p}{f_0 \rho_0}, \tag{2.193}$$

whence

$$u_g = -\frac{\partial \psi}{\partial y}, \qquad v_g = \frac{\partial \psi}{\partial x}. \tag{2.194}$$

The vertical component of vorticity, ζ, is then given by

$$\zeta = \mathbf{k} \cdot \nabla \times \boldsymbol{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla_z^2 \psi. \tag{2.195}$$

⋆ If the Coriolis parameter is not constant, then cross-differentiating (2.190) gives, for constant density geostrophic flow,

$$v_g \frac{\partial f}{\partial y} + f \nabla_z \cdot \boldsymbol{u}_g = 0, \tag{2.196}$$

which implies, using mass continuity,

$$\beta v_g = f \frac{\partial w}{\partial z}. \tag{2.197}$$

where $\beta \equiv \partial f / \partial y = 2\Omega \cos \vartheta / a$. This geostrophic vorticity balance is sometimes known as Sverdrup balance, although that expression is better restricted to the case when the vertical velocity results from external agents, and specifically a wind stress, as considered in chapter 14.

2.8.3 Taylor-Proudman effect

If $\beta = 0$, then (2.197) implies that the vertical velocity is not a function of height. In fact, in that case none of the components of velocity vary with height if density is also constant. To show this, in the limit of zero Rossby number we first write the three-dimensional momentum equation as

$$\boldsymbol{f}_0 \times \boldsymbol{v} = -\nabla \phi - \nabla \chi, \tag{2.198}$$

where $\boldsymbol{f}_0 = 2\boldsymbol{\Omega} = 2\Omega \mathbf{k}$, $\phi = p/\rho_0$, and $\nabla \chi$ represents other potential forces. If $\chi = gz$ then the vertical component of this equation represents hydrostatic balance, and the horizontal components represent hydrostatic balance. On taking the curl of this equation, the terms on the right-hand side vanish and the left-hand side becomes

$$(\boldsymbol{f}_0 \cdot \nabla)\boldsymbol{v} - \boldsymbol{f}_0 \nabla \cdot \boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{f}_0 + \boldsymbol{v} \nabla \cdot \boldsymbol{f}_0 = 0. \tag{2.199}$$

But $\nabla \cdot \boldsymbol{v} = 0$ by mass conservation, and because $\boldsymbol{f}_0$ is constant both $\nabla \cdot \boldsymbol{f}_0$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{f}_0$ vanish. Thus

$$(\boldsymbol{f}_0 \cdot \nabla)\boldsymbol{v} = 0, \tag{2.200}$$

which, since $\boldsymbol{f}_0 = f_0\mathbf{k}$, implies $f_0 \partial \boldsymbol{v}/\partial z = 0$, and in particular we have

$$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \frac{\partial w}{\partial z} = 0. \tag{2.201}$$

A different presentation of this argument proceeds as follows. If the flow is exactly in geostrophic and hydrostatic balance then

$$v = \frac{1}{f_0}\frac{\partial \phi}{\partial x}, \qquad u = -\frac{1}{f_0}\frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial z} = -g \tag{2.202a,b,c}$$

Differentiating (2.202a,b) with respect to z, and using (2.202c) yields

$$\frac{\partial v}{\partial z} = \frac{-1}{f_0}\frac{\partial g}{\partial x} = 0, \quad \frac{\partial u}{\partial z} = \frac{1}{f_0}\frac{\partial g}{\partial y} = 0, \tag{2.203}$$

Noting that the geostrophic veclocities are horizontally non-divergent ($\nabla_z \cdot \boldsymbol{u} = 0$), and using mass continuity then gives $\partial w/\partial z = 0$, as before.

If there is a solid horizontal boundary anywhere in the fluid, for example at the surface, then $w = 0$ at that surface and thus $w = 0$ everywhere. Hence the motion occurs in planes that lie perpendicular to the axis of rotation, and the flow is 'two-dimensional.' This result is known as the *Taylor-Proudman effect,* namely that for constant density flow in geostrophic and hydrostatic balance the vertical derivatives of the horizontal and the vertical velocities are zero.[9] At zero Rossby number, if the vertical velocity is zero somewhere in the flow, it is zero everywhere in that vertical column; furthermore, the horizontal flow has no vertical shear, and the fluid moves like a slab. The effects of rotation have provided a *stiffening* of the fluid in the vertical.

In neither the atmosphere nor the ocean do we observe precisely such vertically coherent flow, mainly because of the effects of stratification. However, it is typical of geophysical fluid dynamics that the assumptions underlying a derivation are not fully satisfied, yet there are manifestations of it in real flow. Thus, one might have naïvely expected, because $\partial w/\partial z = -\nabla_z \cdot \boldsymbol{u}$, that the scales of the various variables would be related by $W/H \sim U/L$. However, if the flow is rapidly rotating we expect that the horizontal flow will be in near geostrophic balance and therefore nearly divergence free; thus $\nabla_z \cdot \boldsymbol{u} \ll U/L$, and $W \ll HU/L$.

2.8.4 Thermal wind balance

Thermal wind balance arises by combining the geostrophic and hydrostatic approximations, and this is most easily done in the context of the anelastic (or Boussinesq) equations, or in pressure coordinates. For the anelastic equations, in spherical coordinates geostrophic balance may be written

$$-fv_g = -\frac{\partial \phi}{\partial x} = -\frac{1}{a\cos\vartheta}\frac{\partial \phi}{\partial \lambda} \tag{2.204a}$$

$$f u_g = -\frac{\partial \phi}{\partial y} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta} \tag{2.204b}$$

Combining these with hydrostatic balance, $\partial \phi / \partial z = b$, gives

$$\boxed{\begin{aligned} -f\frac{\partial v_g}{\partial z} &= -\frac{\partial b}{\partial x} = -\frac{1}{a\cos\lambda}\frac{\partial b}{\partial \lambda} \\ f\frac{\partial u_g}{\partial z} &= -\frac{\partial b}{\partial y} = -\frac{1}{a}\frac{\partial b}{\partial \vartheta} \end{aligned}}\,. \tag{2.205a,b}$$

These equations are known as *thermal wind balance,* and the vertical derivative of the geostrophic wind is the thermal wind. In terms of the zonal angular momentum, the second of these equations may be written

$$\frac{\partial m_g}{\partial z} = -\frac{a}{2\Omega\tan\vartheta}\frac{\partial b}{\partial y}, \tag{2.206}$$

where $m_g = (u_g + \Omega a\cos\vartheta)a\cos\vartheta$. Potentially more accurate than geostrophic balance is the so-called cyclostrophic or gradient-wind balance

$$2u\Omega\sin\vartheta + \frac{u^2}{a}\tan\vartheta \approx -\frac{\partial \phi}{\partial y} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta}. \tag{2.207}$$

For large-scale flow this differs significantly from geostrophic balance only very close to the equator. Combining cyclostrophic and hydrostatic balance gives a modified thermal wind relation, and this takes a simple form when expressed in terms of angular momentum, namely

$$\frac{\partial m^2}{\partial z} \approx -\frac{a^3\cos^3\vartheta}{\sin\vartheta}\frac{\partial b}{\partial y}. \tag{2.208}$$

If the density or buoyancy is constant then there is no shear and (2.205) or (2.208) reduce to the Taylor-Proudman result. But suppose that the temperature falls in the polewards direction. Then thermal wind balance implies that the (eastwards) wind will increase with height — just as is observed in the atmosphere! In general a vertical shear of the horizontal wind is associated with a horizontal temperature gradient, and this is one of the most simple and far-reaching effects in geophysical fluid dynamics.

Pressure coordinates

In pressure coordinates geostrophic balance is just

$$\boldsymbol{f} \times \boldsymbol{u}_g = -\nabla_p \Phi \tag{2.209}$$

where Φ is the geopotential and ∇_p is the gradient operator taken at constant pressure. If f if constant, it follows from (2.209) that the geostrophic wind is nondivergent on pressure surfaces. Taking the vertical derivative of (2.209) (that is, its derivative with respect to p) and using the hydrostatic equation, $\partial\Phi/\partial p = -\alpha$, gives the thermal wind equation

$$\boldsymbol{f} \times \frac{\partial \boldsymbol{u}_g}{\partial p} = \nabla_p \alpha = \frac{R}{p}\nabla_p T, \tag{2.210}$$