

ATMOSPHERIC AND OCEANIC FLUID DYNAMICS

FUNDAMENTALS AND LARGE-SCALE CIRCULATION



Geoffrey K. Vallis

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Vallis, G K. 2005. Atmospheric and Oceanic Fluid Dynamics. Available from www.princeton.edu/~gkv/aofd. (To be published by Cambridge University Press.)

Last updated on September 22, 2005.

The cover is a photograph from space of eddies in the Oyashio Current, the western boundary current of the North Pacific subpolar gyre, off the Kamchatka Peninsula, in March, 1992. The flow is baroclinically unstable, and sea-ice provides flow visualization. Snow cover (white) covers the thicker ice flows just to the right of the center of the image. Courtesy of NASA.

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An asterisk indicates more advanced, but usually uncontroversial, material that may be omitted on a first reading. A dagger indicates material that is still a topic of research or that is not settled. *Caveat emptor.*

We must be ignorant of much, if we would know anything.
Cardinal John Newman (1801–1890).

Preface

THESE ARE MY INCOMPLETE NOTES on the fluid dynamics of the atmosphere and ocean, which is part of the larger field of geophysical fluid dynamics (GFD). It is an incomplete draft, it almost certainly contains errors, and many sources are not properly referenced. There are four parts to the book: basic GFD (chapters 1–5); waves, instabilities and turbulence (chaps 6–10); atmospheric circulation (chaps 11–13); and ocean circulation (chaps 14–16). The concentration in these notes is on the fluid dynamics relevant to the large-scale, extra-tropical general circulation of the atmosphere and ocean, typically involving scales of motion at and larger than the first baroclinic radius of deformation. Furthermore, the chapters on the circulation focus on the steady and statistically steady circulation and perforce a number of important topics are omitted — climate variability, equatorial phenomena, the spin-up of the ocean circulation, the quasi-biennial oscillation. Thus, although the book may appear rather large it is focussed on a limited and hopefully coherent range of material.

I have tried to keep the overall treatment of topics as straightforward and as clear as possible and in practice this means that the level should be appropriate for graduate students. There is a fair amount of repetition in these notes, and some (but not all) of this will be cleaned up as things progress. However, I think that some books have too *little* repetition, so that important things appear once, just like many unimportant things. Repetition can serve both to emphasize the important things, and to keep chapters and sections reasonably self-contained so the reader can jump around if s/he wishes. Obviously the chapters are intellectually linked — for example, heat transport in the atmosphere depends on baroclinic instability, but hopefully the reader already familiar with the latter will be able to read about the former without too much cross-referencing, and will regard the repetition that is present as a ‘feature’ and not a ‘bug’!

The treatment generally is fairly physical and phenomenological, and rigour in the mathematical sense is absent — we treat the derivatives of integrals and of infinitesimal quantities rather informally, for example. An asterisk, *, next to a

section means that it may be omitted on first reading; it may be a little more advanced and is not essential for most of the subsequent material. A dagger, †, next to section means that the section discusses topics of research and may be controversial or even wrong. Roughly speaking, an asterisk typically indicates there is more advanced *manipulation* of the equations, whereas a dagger typically indicates there is *approximation* of the equations. If the asterisk or dagger is applied to a section it applies to all the subsections within. and if a dagger or asterisk appears within a section that is already so-marked, the warnings are even more emphatic. Problems marked with diamonds may be difficult, and I do not know the solutions to all of them. Good answers to some of them are probably publishable and I would appreciate hearing about any such work. Qui docet discit.

Please send your comments, questions and criticism to me at gkv@princeton.edu.

Acknowledgements

NOTATION

Mathematical style generally follows ISO standard practice, as described for example by Beccari (1999). Variables are normally set in italics, differential operators in roman, vectors in bold italics, and tensors in slanting sans serif. A subscript denotes a derivative only if the subscript is a coordinate, such as x , y or z ; a subscript 0 generally denotes a constant reference value (e.g., ρ_0). The components of a vector are denoted by superscripts. The list below contains only the more important variables, or instances of non-obvious notation. Distinct meanings are separated with a semi-colon.

Variable	Description
x, y, z	Cartesian coordinates, usually in zonal, meridional and vertical directions
t	Time
ϑ, λ	Latitude, longitude
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors in (x, y, z) directions
p	Pressure
Z	Log-pressure, $-H \log p/p_0$
L, H	Horizontal length scale, vertical (height) scale
S	Salinity; source term on right-hand side of evolution equation
T	Temperature
θ	Potential temperature
ϕ	Pressure divided by density, p/ρ ; passive tracer
b	Buoyancy, $-g\delta\rho/\rho_0$ or $-g\theta'/\theta_0$
u, v, z	Components of velocity in x, y, z or λ, ϑ, z directions
\mathbf{v}	Three-dimensional velocity, (u, v, z)
\mathbf{u}	Two-dimensional, horizontal velocity (u, v)
$\boldsymbol{\omega}$	Vorticity
ζ	Vertical component of vorticity
ψ	Streamfunction
m	Angular momentum about the earth's axis of rotation
$\Omega, \boldsymbol{\Omega}$	Rotation rate of earth and associated vector
f, f_0	Coriolis parameter, and its reference value
β	Rate of change of f with latitude, $\partial f/\partial\vartheta$
β_T, β_S	Coefficient of expansion with respect to temperature, salinity
N	Buoyancy, or Brunt-Väisälä, frequency
Q	Potential vorticity (in particular Ertel PV)
q	Quasi-geostrophic potential vorticity
\dot{Q}	Rate of heating
\mathcal{F}	Eliassen Palm flux, $(\mathcal{F}^y, \mathcal{F}^z)$
\mathcal{A}	Wave activity
$\boldsymbol{\tau}$	Wind stress
τ	Zonal component or magnitude of wind stress; eddy turnover time
α	Inverse density, or specific volume
ρ	Density
ρ_0	Potential density

Variable	Description
Re	Real part of expression; Reynolds number, UL/ν
Ra	Rayleigh number
Ro	Rossby number, U/fL
Pr	Prandtl ratio, f_0/N
h	Layer thickness (in shallow water equations)
σ	Layer thickness (in isentropic coordinates); Prandtl number
η	Specific entropy; perturbation height; enstrophy cascade or dissipation rate
\mathbf{c}_g	group velocity, (c_g^x, c_g^y, c_g^z)
c_p, c_v	Heat capacities at constant pressure and volume
g	Acceleration due to gravity
ν	Kinematic viscosity
v	Meridional component of velocity
κ	Diffusivity; the ratio R/c_p
γ	Vorticity gradient, $\beta - u_{yy}$; the ratio c_p/c_v
ϵ	Generic small parameter ('epsilon')
ε	Cascade or dissipation rate of energy ('varepsilon')
\mathcal{K}	Kolmogorov or Kolomogorov-like constant
$\left(\frac{\partial a}{\partial b}\right)_c$	Derivative of a with respect to b at constant c .
$\left.\frac{\partial a}{\partial b}\right _{a=c}$	Derivative of a with respect to b evaluated at $a = c$.
∇_z	Gradient operator at constant value of z , $\mathbf{i}\partial_x + \mathbf{j}\partial_y$, and similarly for ∇_x, ∇_y
$\nabla_z \cdot$	Divergence operator at constant value of z , i.e., $\partial_x + \partial_y$, and similarly for $\nabla_x \cdot, \nabla_y \cdot$
curl_z	Vertical component of $\nabla \times$ operator, $\text{curl}_z \mathbf{A} = \mathbf{k} \cdot \nabla \times \mathbf{A} = \partial_x A^y - \partial_y A^x$
$\frac{D}{Dt}$	Material derivative (generic)
$\frac{D}{D_3 t}$	Material derivative in three dimensions, for example $\partial/\partial t + \mathbf{v} \cdot \nabla$.
$\frac{D}{D_2 t}$	Material derivative in two dimensions, for example $\partial/\partial t + \mathbf{u} \cdot \nabla$.
$\frac{D}{D_g t}$	Material derivative using geostrophic velocity, for example $\partial/\partial t + \mathbf{u}_g \cdot \nabla$.

Part I

**FUNDAMENTALS OF
GEOPHYSICAL FLUID DYNAMICS**

Are you sitting comfortably? Then I'll begin.
Julia Lang, *Listen With Mother*, BBC radio program, 1950–1982.

CHAPTER 1

Equations of Motion

THIS CHAPTER establishes the fundamental governing equations of motion for a fluid, with particular reference to the fluids of the earth's atmosphere and ocean.¹ Our approach in many places is quite informal, and the treatment of the standard topics of viscosity and pressure is quite brief.

1.1 TIME DERIVATIVES FOR FLUIDS

The equations of motion of fluid mechanics differ from those of rigid-body mechanics because fluids form a continuum, and because fluids flow and deform. Thus, even though both classical solid and fluid media are governed by the same relatively simple physical laws (Newton's laws and the laws of thermodynamics), the expression of these laws differs between the two. To determine the equations of motion for fluids we must clearly establish what the time derivative of some property of a fluid actually means, and that is the subject of this section.

1.1.1 Field and material viewpoints

In solid-body mechanics one is normally concerned with the position and momentum of identifiable objects — the angular velocity of a spinning top or the motions of the planets around the sun are two well-worn examples. The position and velocity of a particular object is then computed as a function of time by formulating equations of the general form

$$\frac{dx_i}{dt} = F(\{x_i\}, t) \tag{1.1}$$

where $\{x_i\}$ is the set of positions and velocities of all the interacting objects and the operator F on the right-hand side is formulated using Newton's laws of motion.

For example, two massive point objects interacting via their gravitational field obey

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = -\frac{Gm_j}{(\mathbf{r}_i - \mathbf{r}_j)^2} \hat{\mathbf{r}}_{i,j}, \quad i = 1, 2; j = 3 - i. \quad (1.2)$$

We thereby predict the positions, \mathbf{r}_i and velocities, \mathbf{v}_i of the objects given their masses, m_i and the gravitational constant G , and where $\hat{\mathbf{r}}_{i,j}$ is a unit vector directed from \mathbf{r}_i to \mathbf{r}_j .

In fluid dynamics such a procedure would lead to an analysis of fluid motions in terms of the positions and momenta of particular fluid elements, each identified by some label, which might simply be their position at an initial time. We call this a *material* point of view, because we are concerned with identifiable pieces of material; it is also sometimes called a *Lagrangian* view (after J.-L. Lagrange). The procedure is perfectly acceptable in principle, and if followed would provide a complete description of the fluid dynamical system. However, from a practical point of view it is much more than we need, and it would be extremely complicated to implement. Instead, for most problems we would like to know what the values of velocity, density and so on are at *fixed points* in space as time passes. (A weather forecast we might care about tells us how warm it will be where we live, and if we are given that we don't particularly care where a fluid parcel comes from.) Since the fluid is a continuum, this knowledge is equivalent to knowing how the fields of the dynamical variables evolve in space and time, and this is often known as the *field* or *Eulerian* viewpoint (after L. Euler).² Thus, whereas in the material view we consider the time evolution of identifiable fluid elements, in the field view we consider the time evolution of the fluid field from a particular frame of reference. That is, we seek evolution equations of the form

$$\frac{\partial}{\partial t} \phi(x, y, z, t) = F, \quad (1.3)$$

where the field $\phi(x, y, z, t)$ is a dynamical variable (e.g., velocity, density, temperature) which gives the value at any point in space-time, and F is some operator to be determined from Newton's laws of motion and appropriate thermodynamic laws.

Although the field viewpoint will turn out to be the most practically useful, the material description is invaluable both in deriving the equations and in the subsequent insight it frequently provides. This is because the important quantities from a fundamental point of view are often those which are associated with a given fluid element: it is these which directly enter Newton's laws of motion and the thermodynamic equations. It is thus important to have a relationship between the rate of change of quantities associated with a given fluid element and the local rate of change of a field. The material or advective derivative provides this relationship.

1.1.2 The material derivative of a fluid element

A *fluid element* is an infinitesimal, indivisible, piece of fluid — effectively a very small fluid parcel. The material derivative is the rate of change of a property (such as temperature, or momentum) of a particular fluid element. It is also known as the 'substantive derivative' (the derivative associated with a parcel of fluid substance),

the ‘advective derivative’ (because the fluid property is being advected), the ‘convective derivative’ (convection is a slightly old-fashioned name for advection, still used in some fields), or the ‘Lagrangian derivative’. It is just the total time derivative of a property of a piece of fluid.

Let us suppose that a fluid is characterized by a (given) velocity field $\mathbf{v}(\mathbf{x}, t)$, which determines its velocity throughout. Let us also suppose that it has another property ϕ , and let us seek an expression for the rate of change of ϕ of a fluid element. Since ϕ is changing in time and in space we use the chain rule:

$$\delta\phi = \frac{\partial\phi}{\partial t}\delta t + \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z = \frac{\partial\phi}{\partial t}\delta t + \delta\mathbf{x} \cdot \nabla\phi. \quad (1.4)$$

This is true in general for any δt , δx , etc. Thus the total time derivative is

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla\phi. \quad (1.5)$$

If this is to be a material derivative we must identify the time derivative in the second term on the right-hand side with the rate of change of position of a fluid element, namely its velocity. Hence, the material derivative of the property ϕ is

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \mathbf{v} \cdot \nabla\phi. \quad (1.6)$$

The right-hand side expresses the material derivative in terms of the local rate of change of ϕ ($\partial\phi/\partial t$), and a contribution arising from the spatial variation of ϕ , experienced only as the fluid parcel moves. Because the material derivative is so common, and to distinguish it from other derivatives, we denote it by the operator D/Dt . Thus, the material derivative of the field ϕ is

$$\boxed{\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + (\mathbf{v} \cdot \nabla)\phi}. \quad (1.7)$$

The brackets in the last term of this equation are helpful in reminding us that $(\mathbf{v} \cdot \nabla)$ is an operator acting on ϕ .

Material derivative of vector field

The material derivative may act on a vector field \mathbf{b} , in which case

$$\frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{b}. \quad (1.8)$$

In Cartesian coordinates this is

$$\frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + u\frac{\partial\mathbf{b}}{\partial x} + v\frac{\partial\mathbf{b}}{\partial y} + w\frac{\partial\mathbf{b}}{\partial z}, \quad (1.9)$$

and for a particular component of \mathbf{b} ,

$$\frac{Db_x}{Dt} = \frac{\partial b_x}{\partial t} + u\frac{\partial b_x}{\partial x} + v\frac{\partial b_x}{\partial y} + w\frac{\partial b_x}{\partial z}, \quad (1.10)$$

or, in Cartesian tensor notation,

$$\frac{Db_i}{Dt} = \frac{\partial b_i}{\partial t} + v_j \frac{\partial b_i}{\partial x_j} = \frac{\partial b_i}{\partial t} + v_j \partial_j b_i. \quad (1.11)$$

where the subscripts denote the Cartesian components and repeated indices are summed. In coordinate systems other than Cartesian the advective derivative of a vector is not simply the sum of the advective derivative of its components, because the coordinate vectors change direction with position; this will be important when we deal with spherical coordinates (and see problem 2.5). Finally, we note that the advective derivative of the position of a fluid element, \mathbf{r} say, is its velocity, and this may easily be checked by explicitly evaluating $D\mathbf{r}/Dt$.

1.1.3 Material derivative of a finite volume

The volume that a given, unchanging, mass of fluid occupies is deformed and advected by the fluid motion, and there is no particular reason why it should remain constant. Indeed, the volume will change as a result of the movement of each element of its bounding material surface, and will in general change if there is a non-zero normal component of the velocity at the fluid surface. That is, if the volume of some fluid is $\int dV$, then

$$\frac{D}{Dt} \int_V dV = \int_S \mathbf{v} \cdot d\mathbf{S}, \quad (1.12)$$

where the subscript V indicates that the integral is a definite integral over some finite volume V , although the limits of the integral will be functions of time if the volume is changing. The integral on the right-hand side is over the closed surface, S , bounding the volume. Although intuitively apparent (to some), this expression may be derived more formally using Leibnitz's formula for the rate of change of an integral whose limits are changing (problem 1.2). Using the divergence theorem on the right-hand side, (1.12) becomes

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \mathbf{v} dV. \quad (1.13)$$

The rate of change of the volume of an infinitesimal fluid element of volume ΔV is obtained by taking the limit of this expression as the volume tends to zero, giving

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \frac{D\Delta V}{Dt} = \nabla \cdot \mathbf{v}. \quad (1.14)$$

We will often write such expressions informally as

$$\frac{D\Delta V}{Dt} = \Delta V \nabla \cdot \mathbf{v}, \quad (1.15)$$

with the limit implied.

Consider now the material derivative of a property, ϕ , of an infinitesimal fluid element; that is, ϕ is the amount per unit volume of ϕ -stuff — it might, for example, be mass density or density of a dye, per unit volume, in a fluid, and ϕ might be a vector. Then we have

$$\frac{D}{Dt} (\phi \Delta V) = \phi \frac{D\Delta V}{Dt} + \Delta V \frac{D\phi}{Dt}. \quad (1.16)$$

Using (1.15) this becomes

$$\frac{D}{Dt}(\phi\Delta V) = \Delta V \left(\phi \nabla \cdot \mathbf{v} + \frac{D\phi}{Dt} \right), \quad (1.17)$$

and the analogous result for a finite fluid volume is just

$$\frac{D}{Dt} \int_V \phi \, dV = \int_V \left(\phi \nabla \cdot \mathbf{v} + \frac{D\phi}{Dt} \right) dV. \quad (1.18)$$

This expression is to be contrasted with the Eulerian derivative for which the volume, and so the limits of integration, are fixed and we have

$$\frac{d}{dt} \int_V \phi \, dV = \int_V \frac{\partial \phi}{\partial t} \, dV. \quad (1.19)$$

Now consider the material derivative of the integral of fluid property χ multiplied by the mass density, ρ . This arises, for example, in the derivation of the momentum equation, where the momentum of a fluid element is $\rho \mathbf{v} \Delta V$. In general the material derivative of $\chi \rho \Delta V$ is given by

$$\frac{D}{Dt}(\chi \rho \Delta V) = \rho \Delta V \frac{D\chi}{Dt} + \chi \frac{D}{Dt}(\rho \Delta V) \quad (1.20)$$

But $\rho \Delta V$ is just the mass of the fluid element, and that is constant — it is how a fluid element is defined. Thus the second term on the right-hand side vanishes and

$$\frac{D}{Dt}(\chi \rho \Delta V) = \rho \Delta V \frac{D\chi}{Dt} \quad \text{and} \quad \frac{D}{Dt} \int_V \chi \rho \, dV = \int_V \rho \frac{D\chi}{Dt} \, dV, \quad (1.21a,b)$$

where (1.21b) applies to a finite volume. That expression may also be derived more formally using Leibnitz's formula for the material derivative of an integral, and the result also holds when χ is a vector. The result is quite different from the corresponding Eulerian derivative, in which the volume is kept fixed; in that case we have:

$$\frac{d}{dt} \int_V \chi \rho \, dV = \int_V \frac{\partial}{\partial t}(\chi \rho) \, dV. \quad (1.22)$$

Various material and Eulerian derivatives are summarized in the shaded box on the following page.

1.2 THE MASS CONTINUITY EQUATION

In classical mechanics mass is absolutely conserved, and in solid-body there is usually no need for any separate equation to explicitly describe its effects. However, in fluid mechanics fluid flows into and away from regions, and fluid density may change, and an equation that explicitly accounts for the flow of mass is one of the 'equations of motion' of the fluid.

1.2.1 An Eulerian derivation

We will first derive the mass conservation equation from an Eulerian point of view; that is to say, our reference frame is fixed in space and the fluid flows through

Material and Eulerian Derivatives

The material derivative of a scalar (ϕ) and a vector (\mathbf{b}) field are given by:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \mathbf{v} \cdot \nabla\phi, \quad \frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{b}. \quad (\text{D.1})$$

Various material derivatives of integrals are:

$$\frac{D}{Dt} \int_V \phi \, dV = \int_V \left(\frac{D\phi}{Dt} + \phi \nabla \cdot \mathbf{v} \right) dV = \int_V \left(\frac{\partial\phi}{\partial t} + \nabla \cdot (\phi\mathbf{v}) \right) dV, \quad (\text{D.2})$$

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \mathbf{v} \, dV, \quad (\text{D.3})$$

$$\frac{D}{Dt} \int_V \rho\phi \, dV = \int_V \rho \frac{D\phi}{Dt} \, dV. \quad (\text{D.4})$$

These formulae also hold if ϕ is a vector. The Eulerian derivative of an integral is:

$$\frac{d}{dt} \int_V \phi \, dV = \int_V \frac{\partial\phi}{\partial t} \, dV, \quad (\text{D.5})$$

so that

$$\frac{d}{dt} \int_V dV = 0 \quad \text{and} \quad \frac{d}{dt} \int_V \rho\phi \, dV = \int_V \frac{\partial\rho\phi}{\partial t} \, dV. \quad (\text{D.6})$$

it. First we give an elementary derivation in Cartesian coordinates (refer to Fig. 1.1). Consider the infinitesimal, rectangular parallelepiped, control volume $\delta V = \delta x \delta y \delta z$ that is fixed in space. Fluid moves into or out of the volume through its surface S , including through its face in y - z plane, of area $\delta A = \delta y \delta z$ at coordinate

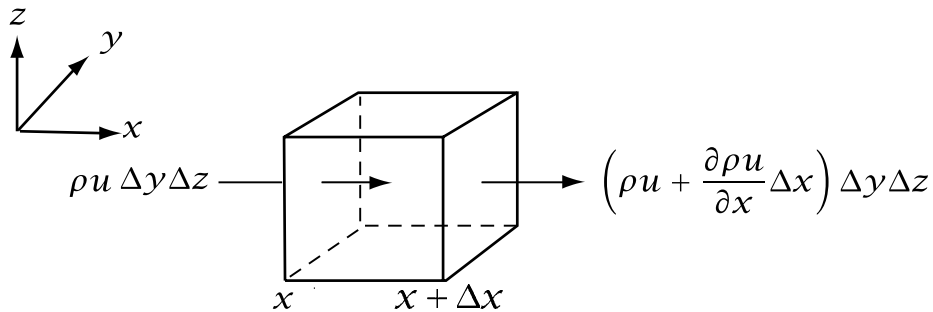


Fig. 1.1 Mass conservation in a cubic Eulerian control volume.

x . The fluid flow through this face, into the control volume, is

$$(\rho u)_x \times \text{Area} = (\rho u)_x \delta y \delta z \quad (1.23)$$

where u is the component of velocity in the x -direction, and the subscript x here denotes the coordinate of the argument. A small distance to the right the flow out of the control volume is

$$(\rho u)_{x+\delta x} \delta y \delta z. \quad (1.24)$$

Thus, the accumulation of fluid within the control volume, due to motion in the x -direction only, is

$$\delta y \delta z [(\rho u)_x - (\rho u)_{x+\delta x}] = -\frac{\partial(\rho u)}{\partial x} \delta x \delta y \delta z. \quad (1.25)$$

To this must be added the effects of motion in the y - and z -directions, namely

$$-\left[\frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z. \quad (1.26)$$

This net accumulation of fluid must be accompanied by a corresponding increase of fluid mass within the control volume. This is

$$\frac{\partial}{\partial t} (\text{Density} \times \text{Volume}) = \delta x \delta y \delta z \frac{\partial \rho}{\partial t}, \quad (1.27)$$

because the volume is constant. Thus, because mass is conserved, (1.25), (1.26) and (1.27) give

$$\delta x \delta y \delta z \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = 0. \quad (1.28)$$

Because the control volume is arbitrary the quantity in square brackets must be zero and we have the *mass continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (1.29)$$

Vector derivation

Consider an arbitrary control volume V bounded by a surface S , fixed in space, with by convention the direction of S being toward the outside of V , as in Fig. 1.2. The rate of fluid loss due to flow through the closed surface S is then given by

$$\text{Fluid loss} = \int_S \rho \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot (\rho \mathbf{v}) dV \quad (1.30)$$

using the divergence theorem. This must be balanced by a change in the mass M of the fluid within the control volume, which, since its volume is fixed, implies a density change. That is

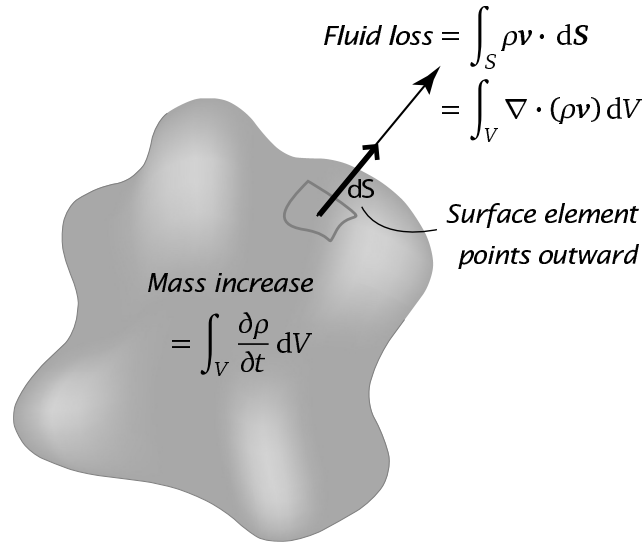
$$\text{Fluid loss} = -\frac{dM}{dt} = -\frac{d}{dt} \int_V \rho dV = -\int_V \frac{\partial \rho}{\partial t} dV. \quad (1.31)$$

Equating (1.30) and (1.31) yields

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0 \quad (1.32)$$

Again, because the volume is arbitrary, the integrand must vanish and we recover (1.29).

Figure 1.2 Mass conservation in an arbitrary Eulerian control volume V bounded by a surface S . The mass gain, $\int_V (\partial\rho/\partial t) dV$ is equal to the mass flowing into the volume, $-\int_S (\rho\mathbf{v}) \cdot d\mathbf{S} = -\int_V \nabla \cdot (\rho\mathbf{v}) dV$.



1.2.2 Mass continuity via the material derivative

We now derive the mass continuity equation (1.29) from a material perspective. This is the most fundamental approach of all since the principle of mass conservation states simply that the mass of a given element of fluid is, by definition of the element, constant. Thus, consider a small mass of fluid of density ρ and volume ΔV . Then conservation of mass may be represented by

$$\frac{D}{Dt}(\rho\Delta V) = 0 \quad (1.33)$$

Both the density and the volume of the parcel may change, so

$$\Delta V \frac{D\rho}{Dt} + \rho \frac{D\Delta V}{Dt} = \Delta V \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right) = 0 \quad (1.34)$$

where the second expression follows using (1.15). Since the volume element is arbitrary, the term in brackets must vanish and

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (1.35)$$

After expansion of the first term this becomes identical to (1.29). This result may be derived more formally by re-writing (1.33) as the integral expression

$$\frac{D}{Dt} \int_V \rho dV = 0. \quad (1.36)$$

Expanding the derivative using (1.18) gives

$$\frac{D}{Dt} \int_V \rho dV = \int_V \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right) dV = 0. \quad (1.37)$$

Because the volume over which the integral is taken is arbitrary the integrand itself must vanish and we recover (1.35). Summarizing, equivalent partial differential

equation representing conservation of mass are:

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0} \quad (1.38a,b)$$

1.2.3 A general continuity equation

The derivation of continuity equation for a general scalar property of a fluid is similar to that for density, except that there may be an external source or sink, and potentially a means of transferring the property from one location to another than by fluid motion, for example by diffusion. If ϕ is the amount of some property of the fluid per unit volume (commonly known as the *concentration* of that property), and if the net effect per unit volume of all nonconservative processes is denoted by $Q[\phi]$, then the continuity equation for concentration may be written:

$$\frac{D}{Dt}(\phi \Delta V) = Q[\phi] \Delta V \quad (1.39)$$

Expanding the left hand side and using (1.15) we obtain

$$\frac{D\phi}{Dt} + \phi \nabla \cdot \mathbf{v} = Q[\phi] \quad (1.40)$$

or equivalently

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}) = Q[\phi]. \quad (1.41)$$

If we are interested in a tracer that is normally measured per unit mass of fluid (which is typical when considering thermodynamic quantities) then the conservation equation would be written

$$\frac{D}{Dt}(\xi \rho \Delta V) = Q[\xi] \rho \Delta V, \quad (1.42)$$

where ξ is the tracer *mixing ratio* — that is, the amount of tracer per unit fluid mass — and $Q[\xi]$ represents nonconservative sources (per unit mass). Then, since $\rho \Delta V$ is constant we obtain

$$\frac{D\xi}{Dt} = Q[\xi]. \quad (1.43)$$

The source term $Q[\xi]$ is evidently equal to the rate of change of ξ of a fluid element. When this is so, it is common to write it simply as $\dot{\xi}$, so that

$$\frac{D\xi}{Dt} = \dot{\xi}. \quad (1.44)$$

A tracer obeying (1.44) with $\dot{\xi} = 0$ is said to be *materially conserved*. If the tracer is materially conserved in the absence of nonconservative sources then it is sometimes said to be ‘semi-materially conserved’.

1.3 THE MOMENTUM EQUATION

The momentum equation is a partial differential equation that describes how the velocity or momentum of a fluid responds to internal and imposed forces. We will derive it using material methods and initially without paying any attention to the those forces, and then we will examine the influences of momentum sources, informally deducing the terms representing the pressure, gravitational and viscous forces.

1.3.1 Advection

Let $\mathbf{m}(x, y, z, t)$ be the momentum-density field (momentum per unit volume) of the fluid. Thus, $\mathbf{m} = \rho \mathbf{v}$ and the total momentum of a volume of fluid is given by the volume integral $\int_V \mathbf{m} dV$. Now, for a fluid the rate of change of a momentum of an identifiable fluid mass is given by the material derivative, and by Newton's second law this is equal to the force acting on it. Thus,

$$\frac{D}{Dt} \int_V \rho \mathbf{v} dV = \int_V \mathbf{F} dV \quad (1.45)$$

Now, using (1.21b) (with χ replaced by \mathbf{v}) the left-hand side is

$$\frac{D}{Dt} \int_V \rho \mathbf{v} dV = \int_V \rho \frac{D\mathbf{v}}{Dt} dV, \quad (1.46)$$

and (1.45) becomes

$$\int_V \left(\rho \frac{D\mathbf{v}}{Dt} - \mathbf{F} \right) dV = 0. \quad (1.47)$$

Because the volume is arbitrary the integrand itself must vanish and we obtain

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F}. \quad (1.48)$$

Expanding the material derivative, this may be written

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{\mathbf{F}}{\rho} \quad (1.49)$$

We have thus obtained an expression for how a fluid accelerates if subject to known forces. These forces are however not all external to the fluid itself; a stress arises from the direct contact between one fluid parcel and another, giving rise to pressure and viscous forces, sometimes referred to as *contact* forces. Because a complete treatment of these would be very lengthy, and is available elsewhere, we treat both of these very informally and intuitively.

1.3.2 The pressure force

Within or at the boundary of a fluid the pressure is the normal force per unit area due to the collective action of molecular motion. Thus

$$d\mathbf{F}_p = -p d\mathbf{S}. \quad (1.50)$$

where p is the pressure, F_p is the pressure force, and dS an infinitesimal surface element. If we grant ourselves this intuitive notion, it is a simple matter to assess the influence of pressure on a fluid, for the pressure force on a volume of fluid is the integral of the pressure over the its boundary and so

$$\mathbf{F}_p = - \int_S p \, d\mathbf{S}. \quad (1.51)$$

The minus sign arises because the pressure force is directed inward, whereas \mathbf{S} is a vector normal to the surface and directed outward. Applying a form of the divergence theorem to the right-hand side gives

$$\mathbf{F}_p = - \int_V \nabla p \, dV \quad (1.52)$$

and so the pressure force per unit volume is just $-\nabla p$. Inserting this into (1.49) we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{\nabla p}{\rho} + \mathbf{F}' \quad (1.53)$$

where \mathbf{F}' includes only viscous and body forces, per unit mass.

1.3.3 Viscosity and diffusion

Viscosity, like pressure, is a force due to the internal motion of molecules. The effects of viscosity are apparent in many situations — the flow of treacle or volcanic lava are obvious examples. In other situations, for example large-scale flow the atmosphere, viscosity is to a first approximation negligible. However, for a constant density fluid viscosity is the *only* way that energy may be removed from the fluid, so that if energy is being added in some way viscosity must ultimately become important if the fluid is to reach an equilibrium where energy input equals energy dissipation. When tea is stirred in a cup, viscosity is the mechanism whereby the fluid eventually stops spinning after we have removed our spoon.

A number of textbooks³ show that, for most Newtonian fluids, the viscous force per unit volume is equal to $\mu \nabla^2 \mathbf{v}$, where μ is a coefficient of diffusivity. Although not exact, this is an extremely good approximation for most liquids and gases. With this term, the momentum equation becomes,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \quad (1.54)$$

where $\nu \equiv \mu/\rho$ is the *kinematic viscosity*. For gases, dimensional arguments suggest that the magnitude of ν should be given by

$$\nu \sim \langle \text{mean free path} \times \text{mean molecular velocity} \rangle \quad (1.55)$$

which for a typical molecular velocity of 300 m s^{-1} and a mean free path of $7 \times 10^{-8} \text{ m}$ gives the not unreasonable estimate of $2.1 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$, within a factor of two of the experimental value (table 1.1). Interestingly, the kinematic viscosity is less for water and mercury than it is for air.

Table 1.1 Experimental values of viscosity for air, water and mercury at room temperature and pressure.

	μ ($\text{kg m}^{-1} \text{s}^{-1}$)	ν ($\text{m}^2 \text{s}^{-1}$)
Air	$1.8 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$
Water	$1.1 \cdot 10^{-3}$	$1.1 \cdot 10^{-6}$
Mercury	$1.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-7}$

1.3.4 Hydrostatic balance

The vertical component — meaning the component parallel to the gravitational force — of the momentum equation is

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (1.56)$$

where w is the vertical component of the velocity. If the fluid is static the gravitational term is balanced by the pressure term and we have

$$\frac{\partial p}{\partial z} = -\rho g, \quad (1.57)$$

and this is known as *hydrostatic balance*. It is clear that the pressure at a point is given by the weight of the fluid above it, provided $p = 0$ at the top of the fluid. It might also appear that (1.57) would be a good *approximation* to (1.56) provided vertical accelerations, Dw/Dt , are small compared to gravity, which is nearly always the case in the atmosphere and ocean. While this statement is true if we need only a reasonably accurate representation of the pressure at a point or in a column, the satisfaction of this condition is not sufficient to ensure that (1.57) provides an accurate enough pressure to determine the *horizontal* pressure gradients responsible for producing motion. We return to this point in section 2.7.

1.4 THE EQUATION OF STATE

In three dimensions the momentum and continuity equations provide four equations, but contain five unknowns — three components of velocity, density and pressure. Obviously other equations are needed, and an *equation of state* relates the various thermodynamic variables to each other. The conventional equation of state is an expression that relates temperature, pressure, composition (salinity), and density, and we may write

$$p = p(\rho, T, S). \quad (1.58)$$

where S is the salinity (see below). An equation of this form is not the most fundamental equation of state from a thermodynamic perspective (an issue we visit later) but it connects readily measurable quantities. For an ideal gas we have

$$p = \rho RT, \quad (1.59)$$

where R is the gas constant for air and T is temperature. (R is related to the universal gas constant R_u by $R = R_u/m$ where m is the mean molecular weight of

the constituents of the gas. Also, $R = nk$ where k is Boltzmann's constant and n is the number of molecules per unit mass.) For dry air, $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$. Air has virtually constant composition, except for variations in water vapour content. A measure of this is the water vapour mixing ratio, $w = \rho_w / \rho_d$ where ρ_w and ρ_d are the densities of water vapour and dry air, respectively, and in the atmosphere w varies between 0 and 0.03. This variation makes the gas constant in the equation of state a weak function of water vapour mixing ratio; that is, $p = \rho R_{\text{eff}} T$ where $R_{\text{eff}} = R_d(1 + wR_v/R_d)/(1 + w)$ where R_d and R_v are the gas constants of dry air and water vapour. Since $w \sim 0.01$ the variation of R_{eff} is quite small and is often ignored, especially in theoretical studies.⁴

For a liquid such as seawater no such analytic expression is easily derivable, and semi-empirical equations are usually resorted to. One complication is the presence of salinity: seawater is in fact a solution of many ions in water — chloride ($\approx 1.9\%$ by weight) sodium (1%), sulfate (0.26%), magnesium (0.13%) and so on, with a total average concentration of about 35‰ (ppt, or parts per thousand). The ratio of the fractions of these salts is more-or-less constant throughout the ocean, and their total concentration may be parameterized by a single measure, the *salinity*, S . Given this, the density of seawater is a function of three variables — pressure, temperature, and salinity. For pure water in a laboratory setting the temperature effect dominates and a reasonable approximation of the equation of state is

$$\rho = \rho_0[1 - \beta_T(T - T_0)], \quad (1.60)$$

where ρ_0 and T_0 are constant reference values, and β_T is a thermal expansion coefficient. Even for pure water, this equation is only valid for small variations around a reference state — note, for example, that pure water generally expands when heated, but that it contracts as its temperature rises from 0°C to 4°C , so that β_T cannot be constant.

In the ocean both pressure and salinity affect the density of seawater and the conventional equation of state has the general form $\alpha = \alpha(T, S, p)$, where $\alpha = 1/\rho$ is the specific volume and S the salinity. Small variations around a reference value can be obtained by using

$$\begin{aligned} d\alpha &= \left(\frac{\partial \alpha}{\partial T}\right)_{S,p} dT + \left(\frac{\partial \alpha}{\partial S}\right)_{T,p} dS + \left(\frac{\partial \alpha}{\partial p}\right)_{T,S} dp \\ &= \alpha(\beta_T dT - \beta_S dS - \beta_p dp), \end{aligned} \quad (1.61)$$

where the second line serves to define the thermal expansion coefficient β_T , the saline contraction coefficient β_S , and the compressibility coefficient β_p (equal to α divided by the bulk modulus). These are in general not constants, but for small variations around a reference state they may be treated as such and we have

$$\alpha = \alpha_0 \left[1 + \beta_T(T - T_0) - \beta_S(S - S_0) - \beta_p(p - p_0) \right]. \quad (1.62)$$

Typical values of these parameters, with variations typically encountered through the ocean, are: $\beta_T \approx 2 (\pm 1.5) \times 10^{-4} \text{ K}^{-1}$ (values increase with both temperature and pressure), $\beta_S \approx 7.6 (\pm 0.2) \times 10^{-4} \text{ ppt}^{-1}$ (ppt = parts per thousand), $\beta_p \approx 4.1 (\pm 0.5) \times 10^{-10} \text{ Pa}^{-1}$. Since the variations around the mean density are small (1.62) can also be written

$$\rho = \rho_0 \left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_p(p - p_0) \right]. \quad (1.63)$$

A linear equation of state for seawater is *not* accurate enough for quantitative oceanography; the β parameters in (1.62) themselves vary with pressure, temperature and (more weakly) salinity so introducing nonlinearities to the equation. The most important of these are captured by an equation of state of the form

$$\alpha = \alpha_0 \left[1 + \beta_T(1 + \gamma^* p)(T - T_0) + \frac{\beta_T^*}{2}(T - T_0)^2 - \beta_S(S - S_0) - \beta_p(p - p_0) \right]. \quad (1.64)$$

The starred constants β_T^* and γ^* capture the leading nonlinearities: γ^* is the *thermobaric* parameter and β_T^* is the second thermal expansion coefficient. Even this expression has quantitative deficiencies and more complicated semi-empirical formulae are often used if high accuracy is needed.⁵ More discussion is to be found in section 1.8.2.

Clearly, the equation of state introduces, in general, a sixth unknown, temperature, and we will have to introduce another physical principle — the first law of thermodynamics or the principle of energy conservation — to obtain a complete set of equations. However, if the equation of state were such that it linked only density and pressure, without introducing another variable, then the equations would be complete; the simplest cases of all being constant density fluids for which the equation of state is just $\rho = \text{constant}$. A fluid for which the density is a function of pressure alone is called a *barotropic fluid*; otherwise, it is a *baroclinic fluid*. Equations of state of the form $p = C\rho^\gamma$ are sometimes called polytropic.

1.5 THE THERMODYNAMIC EQUATION

The thermodynamic equation embodies the principle of the conservation of energy, and in fluids in which the equation of state involves temperature (and not just density and pressure) then the thermodynamic equation is necessary to obtain a closed system of equations.⁶

1.5.1 A few fundamentals

A fundamental postulate of thermodynamics is that the internal energy of a system in equilibrium is a function of its extensive properties volume, entropy, and the mass of its various constituents. (Extensive means that their value depends of the amount of material present, as opposed to an intensive quantity such as temperature.) For our purposes it is more convenient to divide all of these by the mass of fluid present, so expressing the internal energy per unit mass, I , as a function of the specific volume (or inverse density) $\alpha = \rho^{-1}$, the specific entropy η , and the mass fractions of its various components, or its chemical composition, which we parameterize as its salinity S . Thus we have

$$I = I(\alpha, \eta, S), \quad (1.65a)$$

or an equivalent equation for entropy,

$$\eta = \eta(I, \alpha, S). \quad (1.65b)$$

Given the functional forms on the right-hand sides, either of these constitutes a complete description of the macroscopic state of a system in equilibrium, and we call them the *fundamental equation of state*. The first differential of (1.65a) gives, formally,

$$dI = \frac{\partial I}{\partial \alpha} d\alpha + \frac{\partial I}{\partial \eta} d\eta + \frac{\partial I}{\partial S} dS. \quad (1.66)$$

We will now ascribe physical meaning to these differentials.

Conservation of energy states that the internal energy of a body may change because of work done by or on it, or because of a heat input, or because of a change in its chemical composition. We write this as

$$dI = dQ - dW + dC \quad (1.67)$$

where dW is the work done *by* the body, dQ is the heat input *to* the body, and dC accounts for the change in internal energy caused by a change in its chemical composition (e.g., its salinity). This is the first law of thermodynamics. It is applicable to a definite fluid mass, so we can regard dI as the change in internal energy per unit mass, and similarly for the other quantities. Let us consider the causes of variations in these quantities.

Heat Input: The heat input dQ is not the differential of any quantity, and we cannot unambiguously define the heat content of a body as a function of its state. However, the second law of thermodynamics provides a relationship between the heat input and the change in the entropy of a body, namely that in an (infinitesimal) quasi-static or reversible process, with constant composition,

$$Td\eta = dQ, \quad (1.68)$$

where η is the *entropy* of the body. The entropy *is* a function of the state of a body and is, by definition, an adiabatic invariant. It is an extensive quantity, meaning that if we double the amount of material then we double the entropy. Here we will be dealing with the amount of a quantity per unit mass, so that η is the specific entropy.

Work done: The work done by a body is equal to the pressure times the change in its volume. That is

$$dW = p d\alpha, \quad (1.69)$$

where $\alpha = 1/\rho$ is the specific volume of the fluid and p is the pressure.

Composition: The change in internal energy due to compositional changes is related to the change in salinity by

$$dC = \mu dS, \quad (1.70)$$

where μ is the *chemical potential* of the solution. The salinity of a parcel of fluid is conserved unless there are explicit sources and sinks, such as precipitation and evaporation at the surface and molecular diffusion. When these effects do occur the internal energy of a fluid parcel changes by (1.70). However, these effects are usually small, and most important effect

of salinity is that it changes the density of seawater. In the atmosphere the composition of a parcel of air primarily varies according to the amount of water vapour in it; however, the main importance of water vapour is that when condensation or evaporation occurs, heat is released (or required) which provides an entropy source in (1.68).

Collecting equations (1.67) - (1.70) together we have

$$\boxed{dI = T d\eta - p d\alpha + \mu dS} . \quad (1.71)$$

We refer to this (often with $dS = 0$) as *the fundamental thermodynamic relation*. The fundamental equation of state, (1.65), describes the properties of a particular fluid, and the fundamental relation, (1.71), expresses the conservation of energy. Much of classical thermodynamics follows from these two expressions.

1.5.2 * More thermodynamic relations

From (1.71) it follows that

$$T = \left(\frac{\partial I}{\partial \eta} \right)_{\alpha, S}, \quad p = - \left(\frac{\partial I}{\partial \alpha} \right)_{\eta, S}, \quad \mu = \left(\frac{\partial I}{\partial S} \right)_{\eta, \alpha}. \quad (1.72a,b,c)$$

These may be regarded as the defining relations for these variables; it is because of the use of (1.71), and not just the formal expression (1.66), that the pressure and temperature defined this way are indeed related to the internal motion of motion of the molecules that constitute the fluid. Note that if we write

$$d\eta = \frac{1}{T} dI + \frac{p}{T} d\alpha - \frac{\mu}{T} dS, \quad (1.73)$$

it is also clear that

$$p = T \left(\frac{\partial \eta}{\partial \alpha} \right)_{I, S}, \quad T^{-1} = \left(\frac{\partial \eta}{\partial I} \right)_{\alpha, S}, \quad \mu = -T \left(\frac{\partial \eta}{\partial S} \right)_{I, \alpha}. \quad (1.74a,b,c)$$

In the following derivations, we will unless noted suppose that the composition of a fluid parcel is fixed, and drop the suffix S on partial derivatives unless ambiguity might arise.

Because the right-hand-side of (1.71) is equal to an exact differential, the second derivatives are independent of the order of differentiation. That is,

$$\frac{\partial^2 I}{\partial \eta \partial \alpha} = \frac{\partial^2 I}{\partial \alpha \partial \eta} \quad (1.75)$$

and therefore, using (1.72)

$$\left(\frac{\partial T}{\partial \alpha} \right)_{\eta} = - \left(\frac{\partial p}{\partial \eta} \right)_{\alpha}. \quad (1.76)$$

This is one of the *Maxwell relations*, which are a collection of four similar relations which follow directly from the fundamental thermodynamic relation (1.71) and simple relations between second derivatives. A couple of others will be useful.

Define the *enthalpy* of a fluid by

$$h \equiv I + p\alpha \quad (1.77)$$

then, for a parcel of constant composition, (1.71) becomes

$$dh = Td\eta + \alpha dp. \quad (1.78)$$

But h is a function only of η and p so that in general

$$dh = \left(\frac{\partial h}{\partial \eta}\right)_p d\eta + \left(\frac{\partial h}{\partial p}\right)_\eta dp. \quad (1.79)$$

Comparing the last two equations we have

$$T = \left(\frac{\partial h}{\partial \eta}\right)_p \quad \text{and} \quad \alpha = \left(\frac{\partial h}{\partial p}\right)_\eta. \quad (1.80)$$

Noting that

$$\frac{\partial^2 h}{\partial \eta \partial p} = \frac{\partial^2 h}{\partial p \partial \eta} \quad (1.81)$$

we evidently must have

$$\left(\frac{\partial T}{\partial p}\right)_\eta = \left(\frac{\partial \alpha}{\partial \eta}\right)_p, \quad (1.82)$$

and this is our second Maxwell relation.

To obtain the third, we write

$$dI = Td\eta - p d\alpha = d(T\eta) - \eta dT - d(p\alpha) + \alpha dp, \quad (1.83)$$

or

$$dG = -\eta dT + \alpha dp, \quad (1.84)$$

where $G \equiv I - T\eta + p\alpha$ is called the ‘Gibbs free energy’. Now, formally, we have

$$dG = \left(\frac{\partial G}{\partial T}\right)_p dT + \left(\frac{\partial G}{\partial p}\right)_T dp. \quad (1.85)$$

Comparing the last two equations we see that $\eta = -(\partial G/\partial T)_p$ and $\alpha = (\partial G/\partial p)_T$. Furthermore, because

$$\frac{\partial^2 G}{\partial p \partial T} = \frac{\partial^2 G}{\partial T \partial p} \quad (1.86)$$

we have our third Maxwell equation,

$$\left(\frac{\partial \eta}{\partial p}\right)_T = -\left(\frac{\partial \alpha}{\partial T}\right)_p. \quad (1.87)$$

The fourth Maxwell equation, whose derivation is left to the reader, is

$$\left(\frac{\partial \eta}{\partial \alpha}\right)_T = \left(\frac{\partial p}{\partial T}\right)_\alpha, \quad (1.88)$$

and all four Maxwell equations are summarized in the box at the top of the next page. All of them follow from the fundamental thermodynamic relation, (1.71), which is the real silver hammer of thermodynamics.

Maxwell's Relations

The four Maxwell equations are:

$$\begin{aligned} \left(\frac{\partial T}{\partial \alpha}\right)_\eta &= -\left(\frac{\partial p}{\partial \eta}\right)_\alpha, & \left(\frac{\partial T}{\partial p}\right)_\eta &= \left(\frac{\partial \alpha}{\partial \eta}\right)_p, \\ \left(\frac{\partial \eta}{\partial p}\right)_T &= -\left(\frac{\partial \alpha}{\partial T}\right)_p, & \left(\frac{\partial \eta}{\partial \alpha}\right)_T &= \left(\frac{\partial p}{\partial T}\right)_\alpha. \end{aligned} \quad (\text{M.1})$$

These imply:

$$\frac{\partial(T, \eta)}{\partial(p, \alpha)} \equiv \left(\frac{\partial T}{\partial p}\right) \left(\frac{\partial \eta}{\partial \alpha}\right) - \left(\frac{\partial T}{\partial \alpha}\right) \left(\frac{\partial \eta}{\partial p}\right) = 0. \quad (\text{M.2})$$

* Equation of state revisited

The fundamental equation of state (1.65) gives complete information about a fluid in thermodynamic equilibrium, and given this we can obtain expressions for the temperature, pressure and chemical potential using (1.72). These are also equations of state; however, each of them contains less information than the fundamental equation because a derivative has been taken, although all three together provide the same information. Equivalent to the fundamental equation of state are, using (1.78), an expression for the enthalpy as a function of pressure, entropy and composition, or, using (1.84) the Gibbs function as a function of pressure, temperature and composition. The conventional equation of state, (1.58), is obtained by eliminating entropy between (1.72a) and (1.72b). Given the fundamental equation of state, the thermodynamic state of a body is fully specified by a knowledge of any two of p, ρ, T, η and I , plus its composition.

One simple fundamental equation of state is to take the internal energy to be a function of density and not entropy; that is, $I = I(\alpha)$. Bodies with such a property are called *homentropic*. Using (1.72) temperature and chemical potential have no role and the density is a function of pressure alone — the defining property of a barotropic fluid. Neither water nor air are, in general, homentropic but under some circumstances the flow may be adiabatic and $p = p(\rho)$ (e.g., problem 1.10).

In an ideal gas the molecules do not interact except by elastic collisions, and the volume of the molecules is negligible compared to the total volume they occupy. The internal energy then depends only on temperature, and not on the density. A *simple* ideal gas is an ideal gas for which the heat capacity is constant, so that

$$I = cT, \quad (1.89)$$

where c is a constant. Using this and the conventional ideal gas equation, $p = \rho RT$, where R is also constant, we can infer the fundamental equation of state; however, we will defer that until we discuss potential temperature in section 1.5.4. A *general* ideal gas also obeys $p = \rho RT$, but it has heat capacities that may be a function of temperature (but only of temperature — see problem 1.12).

Internal energy and specific heats

We can obtain some useful relations between the internal energy and specific heat capacities, and some useful estimates of their values, by some simple manipulations of the fundamental thermodynamic relation. Assuming that the composition of the fluid is constant (1.71) is

$$T d\eta = dI + p d\alpha, \quad (1.90)$$

so that

$$T d\eta = \left(\frac{\partial I}{\partial T}\right)_\alpha dT + \left[\left(\frac{\partial I}{\partial \alpha}\right)_T + p\right] d\alpha. \quad (1.91)$$

From this, we see that the heat capacity at constant volume (or constant α) c_v is given by

$$c_v \equiv T \left(\frac{\partial \eta}{\partial T}\right)_\alpha = \left(\frac{\partial I}{\partial T}\right)_\alpha. \quad (1.92)$$

Thus, c in (1.89) is equal to c_v .

Similarly, using (1.78) we have

$$\begin{aligned} T d\eta &= dh - \alpha dp \\ &= \left(\frac{\partial h}{\partial T}\right)_p dT + \left[\left(\frac{\partial h}{\partial p}\right)_T - \alpha\right] dp. \end{aligned} \quad (1.93)$$

The heat capacity at constant pressure, c_p , is then given by

$$c_p \equiv T \left(\frac{\partial \eta}{\partial T}\right)_p = \left(\frac{\partial h}{\partial T}\right)_p. \quad (1.94)$$

For later use, we define the ratios $\gamma \equiv c_p/c_v$ and $\kappa \equiv R/c_p$.

For an ideal gas $h = I + RT = T(c_v + R)$. But $c_p = (\partial h/\partial T)_p$, and hence $c_p = c_v + R$, and $(\gamma - 1)/\gamma = \kappa$. Statistical mechanics tells us that for a simple ideal gas the internal energy is equal to $kT/2$ per molecule, or $RT/2$ per unit mass, for each excited degree of freedom, where k is the Boltzmann constant and R the gas constant. The diatomic molecules N_2 and O_2 that comprise most of our atmosphere have two rotational and three translational degrees of freedom, so that $I \approx 5RT/2$, and so $c_v \approx 5R/2$ and $c_p \approx 7R/2$, both being constants. These are in fact very good approximations to the measured values for the earth's atmosphere, and give $c_p \approx 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$. The internal energy is simply $c_v T$ and the enthalpy is $c_p T$. For a liquid, especially one containing dissolved salts such as seawater, no such simple relations are possible: the heat capacities are functions of the state of the fluid, and the internal energy is a function of pressure (or density) as well as temperature.

1.5.3 Thermodynamic equations for fluids

The thermodynamic relations — for example (1.71) — apply to identifiable bodies or systems; thus, the heat input affects the fluid parcel to which it is applied, and we can apply the material derivative to the above thermodynamic relations to obtain equations of motion for a moving fluid. But in doing so we make two assumptions:

- (i) That locally the fluid is in thermodynamic equilibrium. This means that, although the thermodynamic quantities like temperature, pressure and density vary in space and time, locally they are related by the thermodynamic relations such as the equation of state and Maxwell's relations.
- (ii) That macroscopic fluid motions are reversible and so not entropy producing. Thus, the diabatic term dQ represents the entropy sources — such effects as viscous dissipation of energy, radiation, and conduction — whereas the macroscopic fluid motion itself is adiabatic.

The first point requires that the temperature variation on the macroscopic scales must be slow enough that there can exist a volume that is small compared to the scale of macroscopic variations, so that temperature is effectively constant within it, but that is also sufficiently large to contain enough molecules so that macroscopic variables such as temperature have a proper meaning. Accepting these assumptions, the expression

$$T d\eta = dQ \quad (1.95)$$

implies that we may write

$$T \frac{D\eta}{Dt} = \dot{Q}, \quad (1.96)$$

where \dot{Q} is the total rate of heat input per unit mass. This is a thermodynamic equation of motion of the fluid.

For seawater a full specification of its thermodynamic state requires a knowledge of the salinity S , and this is determined by the conservation equation

$$\frac{DS}{Dt} = \dot{S}, \quad (1.97)$$

where \dot{S} represents effects of evaporation and precipitation at the ocean surface, and molecular diffusion. Somewhat analogously, for atmosphere the thermodynamics involve water vapour whose evolution is given by the conservation of water vapour mixing ratio

$$\frac{Dw}{Dt} = \dot{w} \quad (1.98)$$

where \dot{w} represents the effects of condensation and evaporation. Salt has an important effect on the density of seawater, whereas the effect of water vapour on the density of air is slight.

Equation (1.96) is not a *useful* equation unless the entropy can be related to the other fluid variables, temperature, pressure and density. This can be done using the equation of state and the thermodynamic relations we have derived, and is the subject of the following sections. An ideal gas is the simplest case with which to start.

1.5.4 Thermodynamic equation for an ideal gas

For a fluid parcel of constant composition the fundamental thermodynamic relation is

$$dQ = dI + p d\alpha \quad (1.99)$$

For an ideal gas the internal energy is a function of temperature only and $dI = c_v dT$ (also see problems 1.12 and 1.14), so that

$$dQ = c_v dT + p d\alpha \quad \text{or} \quad dQ = c_p dT - \alpha dp, \quad (1.100a,b)$$

where the second expression is derived using $\alpha = RT/p$ and $c_p - c_v = R$. Forming the material derivative of (1.99) gives the general thermodynamic equation

$$\frac{DI}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q}. \quad (1.101)$$

Similarly, for an ideal gas (1.100a,b) respectively give

$$c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q}, \quad \text{or} \quad c_p \frac{DT}{Dt} - \frac{RT}{p} \frac{Dp}{Dt} = \dot{Q}. \quad (1.102a,b)$$

Although (1.102) are equations in the state variables p , T and/or α , time derivatives act on two variables and this is not convenient for many purposes. Using the mass continuity equation, (1.102a) may be written

$$c_v \frac{DT}{Dt} + p \alpha \nabla \cdot \mathbf{v} = \dot{Q}. \quad (1.103)$$

Alternatively, using the ideal gas equation we may eliminate T in favor of p and α , giving the equivalent equation

$$\frac{Dp}{Dt} + \gamma p \nabla \cdot \mathbf{v} = \dot{Q} \frac{\rho R}{c_v}. \quad (1.104)$$

Potential temperature and potential density

When a fluid parcel changes pressure adiabatically, it will expand or contract and, using (1.100b), its temperature change is determined by

$$c_p dT = \alpha dp. \quad (1.105)$$

As this temperature change occurs is not caused by heating, it is useful to define a temperature-like quantity that changes *only* if diabatic effects are present. To this end, we define the *potential temperature*, θ , to be the temperature that a fluid would have if moved adiabatically to some reference pressure (often taken to be the 1000 hPa, which is close to the pressure at the earth's surface). Thus, in adiabatic flow the potential temperature of a fluid parcel is conserved, essentially by definition, and

$$\frac{D\theta}{Dt} = 0. \quad (1.106)$$

Such an evolution equation holds, for adiabatic flow at constant composition, for the potential temperature of any fluid, but to be useful we must be able to relate θ to the other thermodynamic variables. For an ideal gas we use (1.100b) and the equation of state to write the thermodynamic equation as

$$d\eta = c_p d \ln T - R d \ln p. \quad (1.107)$$

The definition of potential temperature then implies that

$$c_p d \ln \theta = c_p d \ln T - R d \ln p, \quad (1.108)$$

and this is satisfied by

$$\theta = T \left(\frac{p_R}{p} \right)^\kappa \quad (1.109)$$

where p_R is a reference pressure and $\kappa = R/c_p$.

Note that

$$d\eta = c_p \frac{d\theta}{\theta} \quad (1.110)$$

and, if c_p is constant,

$$\eta = c_p \ln \theta. \quad (1.111)$$

Equation (1.110) is in fact a general expression for potential temperature of a fluid parcel of constant composition (see section 1.8.1), but (1.111) applies only if c_p is constant, as in, to a very good approximation, the earth's atmosphere.

Using (1.108), the thermodynamic equation in the presence of heating is simply

$$\boxed{c_p \frac{D\theta}{Dt} = \frac{\theta}{T} \dot{Q}}, \quad (1.112)$$

with θ given by (1.109). Equations (1.103), (1.104) and (1.112) are all equivalent forms of the thermodynamic equation for an ideal gas.

The *potential density*, ρ_θ , is the density that a fluid parcel would have if moved adiabatically and at constant composition to a reference pressure, p_R . If the equation of state is written as $\rho = f(p, T)$ then the potential density is just

$$\rho_\theta = f(p_R, \theta). \quad (1.113)$$

For an ideal gas we therefore have

$$\rho_\theta = \frac{p_R}{R\theta}; \quad (1.114)$$

that is, potential density is proportional to the inverse of potential temperature. We may also write (1.114) as

$$\rho_\theta = \rho \left(\frac{p_R}{p} \right)^{1/\gamma}. \quad (1.115)$$

Finally, for later use we note that for small variations around a reference state manipulation of the ideal gas equation gives

$$\frac{\delta\theta}{\theta} = \frac{\delta T}{T} - \kappa \frac{\delta p}{p} = \frac{1}{\gamma} \frac{\delta p}{p} - \frac{\delta\rho}{\rho}. \quad (1.116)$$

** Potential temperature and the fundamental equation of state*

Eq. (1.111) is closely related to the fundamental equation of state: using $I = c_v T$, (1.109), and the equation of state $p = \rho RT$, we can express the entropy explicitly in terms of the density and the internal energy, to wit

$$\boxed{\eta = c_v \ln I - R \ln \rho + \text{constant}}. \quad (1.117)$$

This is the fundamental equation of state for a simple ideal gas. If we were to *begin* with this, we could straightforwardly derive all the thermodynamic quantities of interest for a simple ideal gas: for example, using (1.74a) we immediately recover $P = \rho RT$, and from (1.74b) we obtain $I = c_v T$. Indeed, (1.117) could be used to define a simple ideal gas, but such an *a priori* definition seems a little unmotivated. Of course the heat capacities must still be determined by experiment or by a kinetic theory — they are not given by the thermodynamics, and (1.117) holds only if they are constant.

1.5.5 * Thermodynamic equation for liquids

For a liquid such as seawater no simple exact equation of state exists. Thus, although (1.112) holds at constant salinity for a liquid by virtue of the definition of potential temperature, an accurate expression relating potential temperature to the other thermodynamic variables is nonlinear, complicated and, to most eyes, uninformative. Yet for both theoretical and modelling work a thermodynamic equation is needed to represent energy conservation, and an equation of state needed to close the system, and one of two approaches is thus generally taken: For most theoretical work and for idealized models a simple analytic but approximate equation of state is used, but in situations where more accuracy is called for, such as quantitative modelling or observational work, an accurate but complex semi-empirical equation of state is used. This section outlines how relatively simple thermodynamic equations may be derived that are adequate in many circumstances, and which illustrate the principles used in deriving more complicated equations.

Thermodynamic equation using pressure and density

If we regard η as a function of pressure and density (and salinity if appropriate) we obtain

$$\begin{aligned} T d\eta &= T \left(\frac{\partial \eta}{\partial \rho} \right)_{p,S} d\rho + T \left(\frac{\partial \eta}{\partial p} \right)_{\rho,S} dp + T \left(\frac{\partial \eta}{\partial S} \right)_{\rho,p} dS \\ &= T \left(\frac{\partial \eta}{\partial \rho} \right)_{p,S} d\rho - T \left(\frac{\partial \eta}{\partial p} \right)_{p,S} \left(\frac{\partial \rho}{\partial p} \right)_{\eta,S} dp + T \left(\frac{\partial \eta}{\partial S} \right)_{\rho,p} dS. \end{aligned} \quad (1.118)$$

From this, and using (1.96) and (1.97), we obtain for a moving fluid

$$T \left(\frac{\partial \eta}{\partial \rho} \right)_{p,S} \frac{D\rho}{Dt} - T \left(\frac{\partial \eta}{\partial p} \right)_{p,S} \left(\frac{\partial \rho}{\partial p} \right)_{\eta,S} \frac{Dp}{Dt} = \dot{Q} - T \left(\frac{\partial \eta}{\partial S} \right)_{\rho,p} \dot{S} \quad (1.119)$$

But $(\partial p / \partial \rho)_{\eta,S} = c_s^2$ where c_s is the speed of sound (see section 1.6). This is a measurable quantity in a fluid, and often nearly constant, and so useful to keep in

an equation. Then the thermodynamic equation may be written in the form

$$\boxed{\frac{D\rho}{Dt} - \frac{1}{c_s^2} \frac{Dp}{Dt} = Q[\rho]} \quad (1.120)$$

where $Q[\rho] = (\partial\rho/\partial\eta)_{p,s}\dot{Q}/T - (\partial\rho/\partial S)_{\rho,p}\dot{S}$ appropriately represents the effects of all diabatic and salinity source terms. This form of the thermodynamic equation is valid for both liquids and gases.

Approximations: The speed of sound in a fluid is related to its compressibility — the less compressible the fluid, the greater the sound speed. In a fluid it is often sufficiently high that the second term in (1.120) can be neglected, and the thermodynamic equation takes the simple form:

$$\frac{D\rho}{Dt} = Q[\rho]. \quad (1.121)$$

This equation is a very good approximation for many laboratory fluids. Note that this equation is a thermodynamic equation, arising from the principle of conservation of energy for a liquid. It is a very different equation from the mass conservation equation, which for compressible fluids is also an evolution equation for density.

In the ocean the enormous pressures resulting from columns of seawater kilometers deep mean that although the the second term in (1.120) may be small, it is not negligible, and a better approximation results if we suppose that the pressure is given by the weight of the fluid above it — the hydrostatic approximation. In this case $dp = -\rho g dz$ and (1.120) becomes

$$\frac{D\rho}{Dt} + \frac{\rho g}{c_s^2} \frac{Dz}{Dt} = Q[\rho]. \quad (1.122)$$

In the second term the height field varies much more than the density field, so a good approximation is to replace ρ by a constant, ρ_0 , in this term only. Taking the speed of sound also to be constant gives

$$\frac{D}{Dt} \left[\rho + \frac{\rho_0 z}{H_\rho} \right] = Q[\rho] \quad (1.123)$$

where

$$H_\rho = c_s^2/g \quad (1.124)$$

is the *density scale height* of the ocean. In water, $c_s \approx 1500 \text{ m s}^{-1}$ so that $H_\rho \approx 200 \text{ km}$. The quantity in square brackets in (1.123) is (in this approximation) the *potential density*, this being the density that a parcel would have if moved adiabatically and with constant composition to the reference height $z = 0$. The density scale height as defined here is due to the mean compressibility (i.e., the change in density with pressure) of seawater and, because sound speed varies only slightly in the ocean, this is nearly a

constant. The adiabatic lapse rate of density is the rate at which the density of a parcel changes when undergoing an adiabatic displacement. From (1.123) it is approximately

$$-\left(\frac{\partial \rho}{\partial z}\right)_\eta \approx \frac{\rho_0 g}{c_s^2} \approx 5 \text{ (kg m}^{-3}\text{)/km} \quad (1.125)$$

so that if a parcel is moved adiabatically from the surface to the deep ocean (5 km depth, say) its density will increase by about 25 kg m^{-3} , a fractional density increase of about 1/40 or 2.5%.

Thermodynamic equation using pressure and temperature

Taking entropy to be a function of pressure and temperature (and salinity if appropriate) we have

$$\begin{aligned} T d\eta &= T \left(\frac{\partial \eta}{\partial T}\right)_{p,S} dT + T \left(\frac{\partial \eta}{\partial p}\right)_{T,S} dp + T \left(\frac{\partial \eta}{\partial S}\right)_{T,p} dS \\ &= c_p dT + T \left(\frac{\partial \eta}{\partial p}\right)_{T,S} dp + T \left(\frac{\partial \eta}{\partial S}\right)_{T,p} dS. \end{aligned} \quad (1.126)$$

For a moving fluid, and using (1.96) and (1.97), this implies,

$$\frac{DT}{Dt} + \frac{T}{c_p} \left(\frac{\partial \eta}{\partial p}\right)_{T,S} \frac{Dp}{Dt} = Q[T]. \quad (1.127)$$

where $Q[T] = \dot{Q}/c_p - Tc_p^{-1}\dot{S}(\partial\eta/\partial S)$ includes the effects of the entropy and saline source terms. Now substitute the Maxwell relation (1.87) in the form

$$\left(\frac{\partial \eta}{\partial p}\right)_T = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_p \quad (1.128)$$

to give

$$\frac{DT}{Dt} + \frac{T}{c_p \rho^2} \left(\frac{\partial \rho}{\partial T}\right)_p \frac{Dp}{Dt} = Q[T], \quad (1.129a)$$

or, equivalently,

$$\frac{DT}{Dt} - \frac{T}{c_p} \left(\frac{\partial \alpha}{\partial T}\right)_p \frac{Dp}{Dt} = Q[T]. \quad (1.129b)$$

The density and temperature are related through a measurable coefficient of thermal expansion β_T where

$$\left(\frac{\partial \rho}{\partial T}\right)_p = -\beta_T \rho \quad (1.130)$$

Equation (1.129) then becomes

$$\boxed{\frac{DT}{Dt} - \frac{\beta_T T}{c_p \rho} \frac{Dp}{Dt} = Q[T]}. \quad (1.131)$$

This is form of the thermodynamic equation is valid for both liquids and gases, and in an ideal gas $\beta_T = 1/T$.

Approximations: Liquids are characterized by a small coefficient of thermal expansion, and it is sometimes acceptable in laboratory fluids to neglect the second term on the left-hand side of (1.131). We then obtain an equation analogous to (1.121), namely

$$\frac{DT}{Dt} = Q[T]. \quad (1.132)$$

This approximation relies on the smallness of the coefficient of thermal expansion. A better approximation is to again suppose that the pressure in (1.131) varies according only to the weight of the fluid above it. Then $dp = -\rho g dz$ and (1.131) becomes

$$\frac{1}{T} \frac{DT}{Dt} + \frac{\beta_T g}{c_p} \frac{Dz}{Dt} = \frac{Q[T]}{T}. \quad (1.133)$$

For small variations of T , and if β_T is nearly constant, this simplifies to

$$\frac{D}{Dt} \left(T + \frac{T_0 z}{H_T} \right) = Q[T] \quad (1.134)$$

where

$$H_T = c_p / (\beta_T g) \quad (1.135)$$

is the *temperature scale height* of the fluid. The quantity $T + T_0 z / H_T$ is (in this approximation) the *potential temperature*, θ , so called because it is the temperature that a fluid at a depth z would have if moved adiabatically to a reference depth, here taken as $z = 0$ — the temperature changing because of the work done by or on the fluid parcel as it expands or is compressed. That is,

$$\theta \approx T + \frac{\beta_T g T_0}{c_p} z \quad (1.136)$$

In seawater, however, the expansion coefficient β_T and c_p are functions of pressure and (1.136) is not good enough for quantitative calculations. With the approximate values for the ocean of $\beta_T \approx 2 \times 10^{-4} \text{ K}^{-1}$ and $c_p \approx 4 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$ we obtain $H_T \approx 2000 \text{ km}$.

The adiabatic lapse rate is rate at which the temperature of a parcel changes in the vertical when undergoing an adiabatic displacement. From (1.133) it is

$$\Gamma_{\text{ad}} = - \left(\frac{\partial T}{\partial z} \right)_\eta = \frac{T g \beta_T}{c_p}. \quad (1.137)$$

In general it is a function of temperature, salinity and pressure, but it is a calculable quantity if β_T is known. With the oceanic values above, it is approximately 0.15 K km^{-1} . Again this is not accurate enough for quantitative oceanography because the expansion coefficient is a function of pressure. Nor is it a good measure of stability, because of the effects of salt.

It is interesting that the scale heights given by (1.124) and (1.135) differ so much. As mentioned, the first is due to the compressibility of seawater

Forms of the Thermodynamic Equation

General form

For a parcel of constant composition the thermodynamic equation is

$$T \frac{D\eta}{Dt} = \dot{Q} \quad \text{or} \quad c_p \frac{D \ln \theta}{Dt} = \frac{1}{T} \dot{Q} \quad (\text{T.1})$$

where η is the entropy, θ is the potential temperature, $c_p \ln \theta = \eta$ and \dot{Q} is the heating rate. Applying the first law of thermodynamics $Td\eta = dI + p d\alpha$ gives:

$$\frac{DI}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q} \quad \text{or} \quad \frac{DI}{Dt} + RT \nabla \cdot \mathbf{v} = \dot{Q} \quad (\text{T.2})$$

where I is the internal energy.

Ideal gas

For an ideal gas $dI = c_v dT$, and the (adiabatic) thermodynamic equation may be written in the following equivalent, exact, forms:

$$\begin{aligned} c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} &= 0, & \frac{Dp}{Dt} + \gamma p \nabla \cdot \mathbf{v} &= 0, \\ c_v \frac{DT}{Dt} + p \alpha \nabla \cdot \mathbf{v} &= 0, & \frac{D\theta}{Dt} &= 0, \end{aligned} \quad (\text{T.3})$$

where $\theta = T(p_R/p)^\kappa$. The two expressions on the second line are usually the most useful in modelling and theoretical work.

Liquids

For liquids we may usefully write the (adiabatic) thermodynamic equation as a conservation equation for potential temperature θ or potential density ρ_{pot} and represent these in terms of other variables. For example:

$$\frac{D\theta}{Dt} = 0, \quad \theta \approx \begin{cases} T & \text{(approximately)} \\ T + (\beta_T g z / c_p) & \text{(with some thermal expansion),} \end{cases} \quad (\text{T.4a})$$

$$\frac{D\rho_{\text{pot}}}{Dt} = 0, \quad \rho_{\text{pot}} \approx \begin{cases} \rho & \text{(very approximately)} \\ \rho + (\rho_0 g z / c_s^2) & \text{(with some compression).} \end{cases} \quad (\text{T.4b})$$

Unlike (T.3) these are not equivalent forms. More accurate semi-empirical expressions that may also include saline effects are often used for quantitative applications.

[and so related to c_s^2 , or β_p in (1.63)] whereas the second is due to the change of density with temperature [β_T in (1.63)], and is the distance over which the the difference between temperature and potential temperature

changes by an amount equal to the temperature itself (i.e., by about 273 K). The two heights differ so much because the value of thermal expansion coefficient is not directly tied to the compressibility — for example, fresh water at 4°C has a zero thermal expansion, and so would have an infinite temperature scale height, but its compressibility differs little from water at 20°C. (See also problem 1.20.)

In the atmosphere the ideal gas relationship gives $\beta_T = 1/T$ and so

$$\Gamma_{\text{ad}} = \frac{g}{c_p} \quad (1.138)$$

which is approximately 10 K km^{-1} . The only approximation involved in deriving this is the use of the hydrostatic relationship.

Thermodynamic equation using density and temperature

Taking entropy to be a function of density and temperature (and salinity if appropriate) we have

$$\begin{aligned} Td\eta &= T \left(\frac{\partial \eta}{\partial T} \right)_{\alpha, S} dT + T \left(\frac{\partial \eta}{\partial \alpha} \right)_{T, S} d\alpha + T \left(\frac{\partial \eta}{\partial S} \right)_{T, \alpha} dS \\ &= c_v dT + T \left(\frac{\partial \eta}{\partial \alpha} \right)_{T, S} d\alpha + T \left(\frac{\partial \eta}{\partial S} \right)_{T, \alpha} dS. \end{aligned} \quad (1.139)$$

For a moving fluid this implies,

$$\frac{DT}{Dt} + \frac{T}{c_v} \left(\frac{\partial \eta}{\partial \alpha} \right)_{T, S} \frac{D\alpha}{Dt} = \frac{\dot{Q}}{c_v}. \quad (1.140)$$

If density is nearly constant, as in many liquids, then the second term in (1.140) is small, and also $c_p \approx c_v$.

The thermodynamic equations for a fluid are summarized on page 30, and the complete equations of motion for a fluid are summarized on page 32. Also, note that for ideal gas (1.120) and (1.131) are exactly equivalent to (1.103) or (1.104) (problem 1.11).

1.6 SOUND WAVES

Full of sound and fury, signifying nothing.

William Shakespeare, *Macbeth*, c. 1606.

We now consider, rather briefly, one of the most common phenomena in fluid dynamics yet one which is relatively unimportant for geophysical fluid dynamics — sound waves. Sound itself is not a meteorologically or oceanographically important phenomenon, except in a few special cases, for the pressure disturbance produced by sound waves is a tiny fraction of the ambient pressure and too small to be of importance for the circulation. For example, the ambient surface pressure in the atmosphere is about 10^5 Pa and variations due to large-scale weather phenomena

The Equations of Motion of a Fluid

For dry air, or for a salt-free liquid, the complete set of equations of motion may be written as follows:

The *mass continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (\text{EOM.1})$$

If density is constant this reduces to $\nabla \cdot \mathbf{v} = 0$.

The *momentum equation*:

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} + \mathbf{F}, \quad (\text{EOM.2})$$

where \mathbf{F} represents the effects of body forces such as gravity and ν is the kinematic viscosity. If density is constant, or pressure is given as a function of density alone (e.g., $p = C\rho^\gamma$), then (EOM.1) and (EOM.2) form a complete system.

The *thermodynamic equation*:

$$\frac{D\theta}{Dt} = \frac{1}{c_p} \left(\frac{\theta}{T} \right) \dot{Q}. \quad (\text{EOM.3})$$

where \dot{Q} represents external heating and diffusion, the latter being $\kappa \nabla^2 \theta$ where κ is the diffusivity.

The *equation of state*:

$$\rho = g(\theta, p) \quad (\text{EOM.4})$$

where g is a given function. For example, for an ideal gas, $\rho = p_R^\kappa / (R\theta p^{\kappa-1})$.

The equations describing fluid motion are called the *Euler equations* if the viscous term is omitted, and the *Navier-Stokes equations* if viscosity is included.⁷ Sometimes the Euler equations are taken to mean the momentum and mass conservation equations for an inviscid fluid of constant density.

are about 10^3 Pa, often larger, whereas sound waves of 60 dB produce pressure variations of about 10 Pa.

The smallness of the disturbance produced by sound waves justifies a linearization of the equations of motion about a spatially uniform basic state (denoted by a subscript 0) that is a time-independent solution to the equations of motion. Thus, we write $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, $\rho = \rho_0 + \rho'$ and so on, substitute in the equations of motion, and neglect terms involving products of primed quantities, as these are assumed small. By choice of our reference frame we will simplify matters further by setting

$\mathbf{v}_0 = 0$. The linearized momentum and mass conservation equations are then

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' \quad (1.141)$$

$$\rho_0 \frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}' \quad (1.142)$$

These linear equations do not in themselves determine the magnitude of the disturbance, and the linear approximation must be checked *a posteriori* (problem 1.5). Now, sound waves are largely adiabatic. Thus,

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial \rho} \right)_\eta \frac{d\rho}{dt}, \quad (1.143)$$

where $(\partial p / \partial \rho)_\eta$ is the derivative at constant entropy, whose particular form is given by the equation of state for the fluid at hand. Then, from (1.141) - (1.143) we obtain a single equation for pressure,

$$\frac{\partial^2 p'}{\partial t^2} = c_s^2 \nabla^2 p', \quad (1.144)$$

where $c_s^2 = (\partial p / \partial \rho)_\eta$. Eq. (1.144) is the classical wave equation; solutions propagate at a speed c_s which thus may be identified as the speed of sound. In an ideal gas manipulation of the equation of state leads to $p = C\rho^\gamma$ for adiabatic compressions, whence $c_s^2 = \gamma p / \rho = \gamma RT$ where $\gamma = c_p / c_v$. Values of γ typically range from 5/3 for a monatomic gas to 7/5 for a diatomic gas. For air, which is almost entirely diatomic, this leads to $c_s \approx 350 \text{ m s}^{-1}$ at 300 K. In seawater no such theoretical approximation is easily available but measurements show that $c_s \approx 1500 \text{ m s}^{-1}$.

1.7 COMPRESSIBLE AND INCOMPRESSIBLE FLOW

There are many important circumstances in fluid dynamics when the dynamics are completely specified by the momentum equation and a form of the mass conservation equation. This arises in general when the equation of state does not introduce temperature, and the simplest such equation of state would be that density be constant. Although there are probably no fluids of exactly constant density, in many cases the density of a fluid will vary so little that it is a very good approximation to consider the density effectively constant in the mass conservation equation. The motion is then said to be incompressible. For example, in the earth's oceans the density varies by less than 5%, even though the pressure at the ocean bottom, a few kilometers below the surface, is several hundred times the atmospheric pressure at the surface.

Let us first note how the mass conservation equation simplifies when density is strictly constant, and then consider under what conditions this is a good approximation.

1.7.1 Constant density fluids

If a fluid is strictly of constant density, the mass continuity equation (1.38) simplifies easily by neglecting all derivatives of density yielding

$$\boxed{\nabla \cdot \mathbf{v} = 0}. \quad (1.145)$$

The *prognostic* equation (1.38) has become a *diagnostic* equation (1.145), or a constraint to be satisfied by the velocity at each instant of the fluid motion. A consequence of this equation is that the volume of each material fluid element remains constant. To see this recall the expression for the conservation of mass in the form

$$\frac{D}{Dt}(\rho \Delta V) = 0. \quad (1.146)$$

If ρ is constant this reduces to an expression for volume conservation, namely $D\Delta V/Dt = 0$, whence (1.145) is recovered because $D\Delta V/Dt = \Delta V \nabla \cdot \mathbf{v}$. The absolute constancy of density is an idealization which is rarely, if ever, realized in the natural world. However, there are many cases in which it is a very good approximation to use (1.145) in place of the full continuity equation. These depend not only on the physical nature of the fluid but also on the flow itself, as we shall now discuss.

1.7.2 Incompressible flows

An incompressible fluid is one in which the density of a given fluid element does not change.⁸ Thus, in the mass continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (1.147)$$

the material derivative of density is neglected and we obtain,

$$\nabla \cdot \mathbf{v} = 0. \quad (1.148)$$

In reality no fluid is truly incompressible and for (1.148) to approximately hold we just require that that

$$|D\rho/Dt| \ll |\rho \nabla \cdot \mathbf{v}|. \quad (1.149)$$

Our working definition of incompressibility, then, is that *in an incompressible fluid density changes (from whatever cause) are so small that they have a negligible affect on the mass balance*, and (1.147) may be replaced by (1.148). We do not need to assume that the densities of differing fluid elements are similar to each other, just that the density of any particular fluid element is nearly constant, but in the ocean (and in most liquids) it is in fact the case that variations in density are everywhere small compared to the mean density, and it this is condition that we generally apply. The atmosphere is not incompressible and (1.148) does not in general hold there. Note also that the fact that $\nabla \cdot \mathbf{v} = 0$ does *not* imply that we may independently use $D\rho/Dt = 0$. Indeed for a liquid with equation of state $\rho = \rho_0(1 - \beta_T(T - T_0))$ and thermodynamic equation $c_p DT/Dt = \dot{Q}$ we obtain

$$\frac{D\rho}{Dt} = -\frac{\beta_T \rho_0}{c_p} \dot{Q}. \quad (1.150)$$

Note too that incompressibility does not necessarily imply the neglect of density variations in the momentum equation — it is only in the mass continuity equation that density variations are neglected.

Some conditions for incompressibility

The condition that density is largely unaffected by pressure gives one necessary condition for the legitimate use of (1.148), as follows. First assume adiabatic flow, and omit the gravitational term. Then

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial \rho} \right)_\eta \frac{d\rho}{dt} = c_s^2 \frac{d\rho}{dt} \quad (1.151)$$

so that the density and pressure variations of a fluid parcel are related by

$$\delta p \sim c_s^2 \delta \rho. \quad (1.152)$$

From the momentum equation we estimate

$$\frac{U^2}{L} \sim \frac{1}{L} \frac{\delta p}{\rho_0}, \quad (1.153)$$

where U and L are typical velocities and lengths and where ρ_0 is a representative value of the density. Using (1.152) and (1.153) gives $U^2 \sim c_s^2 \delta \rho / \rho_0$. The incompressibility condition $\delta \rho / \rho_0 \ll 1$ then becomes

$$\frac{U^2}{c_s^2} \ll 1. \quad (1.154)$$

That is, for a flow to be incompressible the fluid velocities must be less than the speed of sound, or the Mach number, $M \equiv U/c_s$, must be small.

In the earth's atmosphere it is apparent that density changes significantly with height. Assuming hydrostatic balance and an ideal gas, then

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -g, \quad (1.155)$$

and if (for simplicity) we assume that atmosphere is isothermal then

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial \rho} \right)_T \frac{\partial \rho}{\partial z} = RT_0 \frac{\partial \rho}{\partial z}. \quad (1.156)$$

Using (1.155) and (1.156) gives

$$\rho = \rho_0 \exp(-z/H_\rho), \quad (1.157)$$

where $H_\rho = RT_0/g$ is the (density) *scale height* of the atmosphere. It is easy to see that density changes are negligible only if we concern ourselves with motion less than the scale height, so this is another necessary condition for incompressibility.

In the atmosphere, although the Mach number is small for most flows, vertical displacements often exceed the scale height and in those cases the flow cannot be considered incompressible. In the ocean density changes from all causes are small and in most circumstances the ocean may be considered to contain an incompressible fluid. We return to this in the next chapter when we consider the Boussinesq equations.

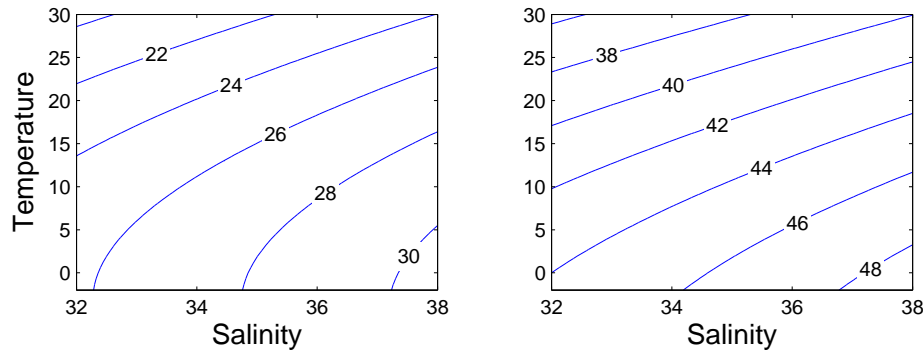


Fig. 1.3 A temperature-salinity diagram for seawater, calculated using an empirical equation of state. Contours are (density - 1000) kg m⁻³, and the temperature is potential temperature, which in the deep ocean may be less than *in situ* temperature a degree or so (see Fig. 1.4). Left panel: at sea-level ($p = 10^5$ Pa = 1000 mb). Right panel: at $p = 4 \times 10^7$ Pa, a depth of about 4 km. Note that in both cases the contours are slightly convex.

1.8 * MORE THERMODYNAMICS OF LIQUIDS

1.8.1 Potential temperature, potential density and entropy

For an ideal gas we were able to derive a single prognostic equation for a single variable, potential temperature. As potential temperature is in turn simply related to the temperature and pressure, this is a useful prognostic equation. Can we achieve something similar with a more general equation of state, with non-constant coefficients of expansion?

Potential temperature

The potential temperature is defined as the temperature that a parcel would have if moved adiabatically to a given reference pressure p_R , often taken as 10^5 Pa (or 1000 hPa, or 1000 mb, approximately the pressure at the sea-surface). Thus it may be calculated, at least in principle, through an integral of the form

$$\theta(S, T, p; p_R) = T + \int_p^{p_R} \Gamma'_{\text{ad}}(S, T, p) dp \quad (1.158)$$

where $\Gamma'_{\text{ad}} = (\partial T / \partial p)_\eta$. The potential temperature of a fluid parcel is directly related to its entropy, provided its composition does not change. We already demonstrated this for an ideal gas, and to see it explicitly in the general case let us first write the equation of state in the form

$$\eta = \eta(S, T, p). \quad (1.159)$$

Now, by definition of potential temperature we have

$$\eta = \eta(S, \theta; p_R) \quad \text{and} \quad \theta = \theta(\eta, S; p_R). \quad (1.160)$$

For a parcel of constant salinity, changes in entropy are caused only by changes in potential temperature so that

$$d\eta = \frac{\partial \eta(S, \theta; p_R)}{\partial \theta} d\theta. \quad (1.161)$$

Now, if we express entropy as a function of temperature and pressure then

$$\begin{aligned} Td\eta &= T \left(\frac{\partial \eta}{\partial T} \right)_p dT + T \left(\frac{\partial \eta}{\partial p} \right)_T dp \\ &= c_p dT - T \left(\frac{\partial \alpha}{\partial T} \right)_p dp. \end{aligned} \quad (1.162)$$

using one of the Maxwell relations. Suppose a fluid parcel moves adiabatically, then $d\eta = 0$ and, by (1.161), $d\theta = 0$. That is, the potential temperature at each point along its trajectory is constant and $\theta = \theta(\eta)$. How do we evaluate this function? Simply note that the temperature at the reference pressure p_R is the potential temperature, so that directly from (1.162)

$$\boxed{d\eta = c_p(p_R, \theta) \frac{d\theta}{\theta}}, \quad (1.163)$$

and $d\eta/d\theta = c_p(p_R, \theta)/\theta$. If c_p is constant this integrates to

$$\eta = c_p \ln \theta + \text{constant} \quad (1.164)$$

as for a simple ideal gas (1.111).

Since potential temperature is conserved in adiabatic motion, the thermodynamic equation can be written

$$c_p \frac{D\theta}{Dt} = \frac{\theta}{T} \dot{Q}. \quad (1.165)$$

where the right-hand-side represents heating. (If salinity is changing, then the right-hand side should also include any saline source terms and saline diffusion. However, such terms usually have a very small effect.) This equation is equivalent to (1.120) or (1.131), although it is only useful if θ can be simply related to the other state variables. In principle this can be done using (1.158), and in practice empirical relationships have been derived that express potential temperature in terms of the local temperature, pressure and salinity, and density in terms of potential temperature, salinity and pressure (see section 1.8.2 for more discussion).

Potential density

Potential density, ρ_θ , is defined as the density that a parcel would have if moved adiabatically and with fixed composition to a given reference pressure p_R often, but not always, taken as 10^5 Pa, or 1 bar. If the equation of state is of the form $\rho = \rho(S, T, p)$ then by definition we have

$$\rho_\theta = \rho(S, \theta; p_R). \quad (1.166)$$

For a parcel moving adiabatically (i.e., fixed salinity and entropy or potential temperature) its potential density is therefore conserved. For an ideal gas (1.166) gives

$\rho_\theta = p_R/(R\theta)$ [as in (1.114)] and potential density provides no more information than potential temperature. However, in the oceans potential density accounts for the effect of salinity on density and so is a much better measure of the static stability of a column of water than density itself.

From (1.123) an approximate expression for the potential density in the ocean is

$$\rho_\theta = \left(\rho + \frac{\rho_0 g z}{c_s^2} \right). \quad (1.167)$$

Although this may suffice for theoretical or some modelling work, the vertical gradient of potential temperature in the ocean is often close to zero and a still more accurate, generally semi-empirical, expression is needed to determine stability properties.

Because density is so nearly constant in the ocean, it is common to subtract the amount 1000 kg m^{-3} before quoting its value, and depending on whether this value refers to *in situ* density or the potential density the results are called σ_t ('sigma-tee') or σ_θ ('sigma-theta') respectively. Thus,

$$\sigma_t = \rho(p, T, S) - 1000, \quad \sigma_\theta = \rho(p_R, \theta, S) - 1000. \quad (1.168a,b)$$

If the potential density is referenced to a particular level, this is denoted by a subscript on the σ . Thus, σ_2 is the potential density referenced to 200 bars of pressure, or about 2 kilometers depth.

1.8.2 * More About Seawater

We now consider, rather didactically, some of the properties of the equation of state for seawater, noting in particular those nonlinearities that, although small, give it somewhat peculiar properties. We use a prototypical equation of state, (1.64) that, although not highly accurate except for small variations around a reference state, does capture the essential nonlinearities.⁹ That equation of state may be written as:

$$\alpha = \alpha_0 \left[1 + \beta_T(1 + \gamma^* p)(T - T_0) + \frac{\beta_T^*}{2}(T - T_0)^2 - \beta_S(S - S_0) - \beta_p(p - p_0) \right], \quad (1.169)$$

where we may take $p_0 = 0$ and $\beta_p = \alpha_0/c_0'^2$, where c_0' is a reference sound speed. The starred parameters are associated with the nonlinear terms: β_T^* is the second expansion coefficient and γ^* is the 'thermobaric parameter', which determines the extent to which the thermal expansion of water depends on temperature. An equation of this form is useful because its coefficients can, in principle, be measured in the field or in the laboratory, and approximate values are given in table 1.2. However, it may not be the most useful form for modelling or observational work, because T is not materially conserved. Let us use this equation to deduce various thermodynamic quantities of interest, and also transform it to a more useful form for modelling.

Potential temperature of seawater

It would be useful to express (1.169) in terms of materially conserved variables, and so in terms of potential temperature rather than temperature. Now, by definition

Parameter	Description	Value
ρ_0	Reference Density	$1.027 \times 10^3 \text{ kg m}^{-3}$
α_0	Reference Specific Volume	$9.738 \times 10^{-4} \text{ m}^3 \text{ kg}^{-1}$
T_0	Reference temperature	283 K
S_0	Reference salinity	35 psu \approx 35‰
c_0	Reference sound speed	1490 m s^{-1}
β_T	First thermal expansion coefficient	$1.67 \times 10^{-4} \text{ K}^{-1}$
β_T^*	Second thermal expansion coefficient	$1.00 \times 10^{-5} \text{ K}^{-2}$
β_S	Haline contraction coefficient	$0.78 \times 10^{-3} \text{ psu}^{-1}$
β_p	Inverse bulk modulus ($\approx \alpha_0/c_0^2$)	$4.39 \times 10^{-10} \text{ m}^2 \text{ kg}^{-1}$
γ^*	Therobaric parameter ($\approx \gamma'^*$)	$1.1 \times 10^{-8} \text{ Pa}^{-1}$
c_{p0}	Specific heat capacity at constant pressure	$3986 \text{ J kg}^{-1} \text{ K}^{-1}$

Table 1.2 Various thermodynamic and equation-of-state parameters for seawater. Specifically, these parameters may be used in the approximate equations of state (1.64) and (1.178).

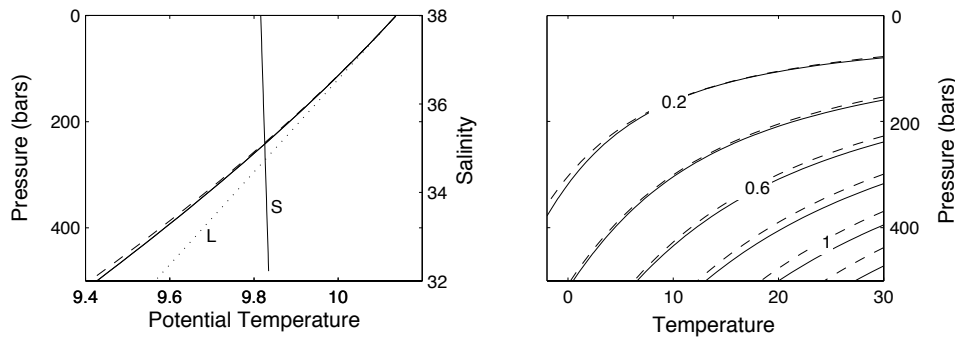


Fig. 1.4 Examples of variation of potential temperature of seawater with pressure, temperature and salinity. Left panel: the sloping lines show potential temperature as a function of pressure at fixed salinity ($S = 35$ psu) and temperature (13.36°C , the value of potential temperature at the surface). The solid line is computed using an accurate, empirical equation of state, the almost-coincident dashed line uses the simpler expression (1.177a) and the dotted line (labelled L) uses the even simpler linear expression (1.177c). The near vertical solid line, labelled S, shows the variation of potential temperature with salinity at fixed temperature and pressure. Right panel: Contours of the difference between temperature and potential temperature, ($T - \theta$) in the pressure-temperature plane, again for $S = 35$ psu. The solid lines use very accurate empirical formula, and the dashed lines use (1.177). The simpler equation can be improved locally, but not globally, by tuning the coefficients. (100 bars of pressure (10^7 Pa or 10 MPa) is approximately 1 km depth.)

the potential temperature is obtained by integrating the adiabatic lapse rate from the in situ pressure to the reference pressure (zero); that is

$$\theta - T = \int_z^{z(p=0)} \left(\frac{\partial T}{\partial z} \right)_\eta dz = \int_p^0 \left(\frac{\partial T}{\partial p} \right)_\eta dp \quad (1.170)$$

Using (1.162), the adiabatic lapse rate is

$$\left(\frac{\partial T}{\partial p} \right)_\eta = \frac{T}{c_p} \left(\frac{\partial \alpha}{\partial T} \right)_{p,S} = \frac{T}{c_p} \alpha_0 [\beta_T (1 + \gamma^* p) + \beta_T^* (T - T_0)]. \quad (1.171)$$

Now, c_p satisfies $c_p = T(\partial \eta / \partial T)_p$, so that, using the Maxwell relation (1.87),

$$\left(\frac{\partial c_p}{\partial p} \right)_{T,S} = T \left(\frac{\partial}{\partial T} \left(\frac{\partial \eta}{\partial p} \right)_T \right)_p = T \frac{\partial^2 \alpha}{\partial T^2}. \quad (1.172)$$

Thus, for our equation of state, we have

$$\left(\frac{\partial c_p}{\partial p} \right)_{T,S} = -T \alpha_0 \beta_T^*, \quad (1.173)$$

and therefore

$$c_p = c_{p0}(T, S) - p T \alpha_0 \beta_T^*. \quad (1.174)$$

The first term cannot be determined solely from the conventional equation of state; in fact for seawater specific heat varies very little with temperature (of order one part in a thousand for a 10 K temperature variation). It varies more with salinity, changing by about $-5 \text{ J kg}^{-1} \text{ K}^{-1}$ per part-per-thousand change in salinity. Thus we take

$$c_{p0}(T, S) = c_{p1} + c_{p2}(S - S_0), \quad (1.175)$$

where c_{p1} and c_{p2} are constants that may be experimentally determined.

Using (1.174) and (1.171) in (1.170) gives,

$$\theta = T \exp \left\{ - \frac{\alpha_0 \beta_T p}{c_{p0}} \left[1 + \frac{1}{2} \gamma^* p + \frac{\beta_T^*}{\beta_T} (T - T_0) \right] \right\}. \quad (1.176)$$

This equation is a relationship between T , θ and p analogous to (1.109) for an ideal gas. The exponent itself is small, the second and third terms in square brackets are small compared to unity, and the deviations of both T and θ from T_0 are also presumed small. Taking advantage of all of this enables the expression to be rewritten, with increasing levels of approximation, as

$$T' \approx \frac{T_0 \alpha_0 \beta_T}{c_{p0}} p \left(1 + \frac{1}{2} \gamma^* p + T_0 \frac{\alpha_0 \beta_T^*}{c_{p0}} p \right) + \theta' \left(1 + T_0 \frac{\alpha_0 \beta_T^*}{c_{p0}} p \right), \quad (1.177a)$$

$$\approx \frac{T_0 \alpha_0 \beta_T}{c_{p0}} p \left(1 + \frac{1}{2} \gamma^* p \right) + \theta' \left(1 + T_0 \frac{\alpha_0 \beta_T^*}{c_{p0}} p \right), \quad (1.177b)$$

$$\approx \frac{T_0 \alpha_0 \beta_T}{c_{p0}} p + \theta', \quad (1.177c)$$

where $T' = T - T_0$ and $\theta' = \theta - T_0$. The last of the three, (1.177c), holds for a linear

equation of state, and is useful for calculating approximate differences between temperature and potential temperature; making use of the hydrostatic approximation reveals that it is essentially the same as (1.136). Note that the potential temperature is related to temperature via the thermal expansion coefficient and not, as one might naïvely have expected, the compressibility coefficient. Plots of the difference between temperature and potential temperature, that also give both a sense of the accuracy of these simpler formula, is given in Fig. 1.4.

Using (1.177b) in the equation of state (1.169) we find that, approximately,

$$\alpha = \alpha_0 \left[1 - \frac{\alpha_0 p}{c_0^2} + \beta_T (1 + \gamma'^* p) \theta' + \frac{1}{2} \beta_T^* \theta'^2 - \beta_S (S - S_0) \right], \quad (1.178)$$

where $\gamma'^* = \gamma^* + T_0 \beta_T^* \alpha_0 / c_{p0} \approx \gamma^*$ and $c_0'^{-2} = c_0'^{-2} - \beta_T^2 T_0 / c_p \approx c_0'^{-2}$ is a reference value of the speed of sound (γ^* and γ'^* differ by a few percent, and c_0^2 and $c_0'^2$ differ by only a few parts in a thousand). Given (1.178), it is in principle straightforward, although in practice rather tedious, to compute various thermodynamic quantities of interest; a calculation of the buoyancy frequency is given in problem 2.19. We may approximate (1.178) further by using the hydrostatic pressure instead of the actual pressure; thus, letting $p = -g(z - z_0) / \alpha_0$ where z_0 is the nominal value of z at which $p = 0$, we obtain

$$\alpha = \alpha_0 \left[1 + \frac{g(z - z_0)}{c_0^2} + \beta_T \left(1 - \gamma'^* \frac{g(z - z_0)}{\alpha_0} \right) \theta' + \frac{\beta_T^*}{2} \theta'^2 - \beta_S (S - S_0) \right]. \quad (1.179)$$

Using z instead of p in the equation of state entails a slight loss of accuracy, but it turns out to be necessary to ensure that the Boussinesq equations conserve energy and potential vorticity, as discussed in later chapters.

The variation of density with potential temperature and salinity and pressure is illustrated in Fig. 1.3.

1.9 THE ENERGY BUDGET

The total energy of a fluid includes the kinetic, potential and internal energies. Both fluid flow and pressure forces will in general move energy from place to place, but we nevertheless expect energy to be conserved in an enclosed volume. Let us now consider what form energy conservation takes in a fluid.

1.9.1 Constant density fluid

For a constant density fluid the momentum equation and the mass continuity equation $\nabla \cdot \mathbf{v} = 0$, are sufficient to completely determine the evolution of a system. The momentum equation is

$$\frac{D\mathbf{v}}{Dt} = -\nabla(\phi + \Phi) + \nu \nabla^2 \mathbf{v}, \quad (1.180)$$

where $\phi = p / \rho_0$ and Φ is the potential for any conservative force (e.g., gz for a uniform gravitational field). We can rewrite the advective term on the left-hand-side using the identity,

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = -\mathbf{v} \times \boldsymbol{\omega} + \nabla(\mathbf{v}^2 / 2), \quad (1.181)$$

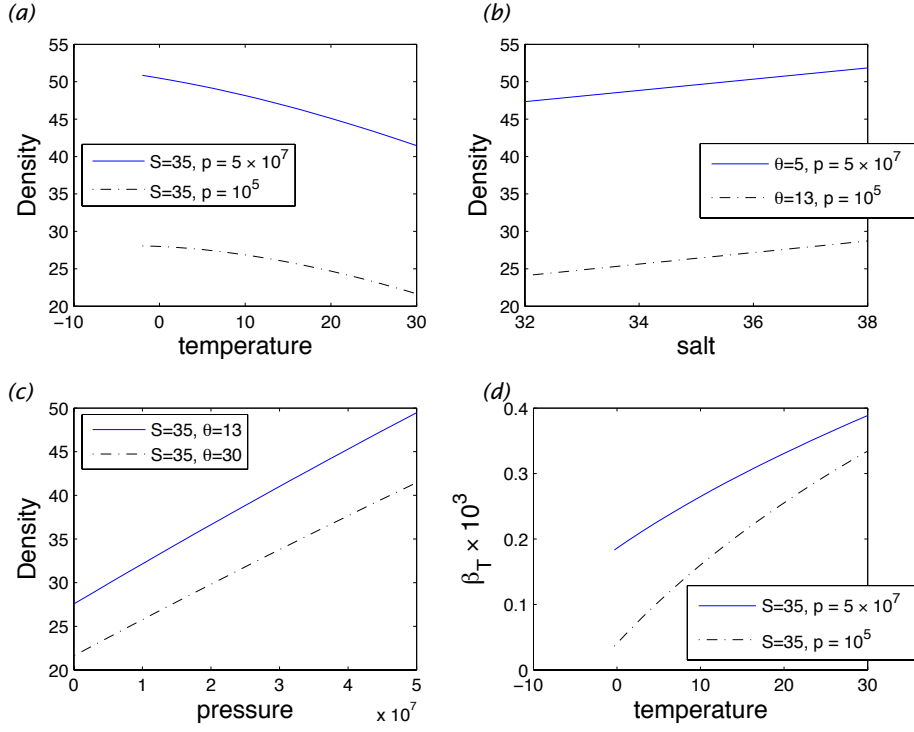


Fig. 1.5 Examples of the variation of density of seawater (minus 1000 kg m^{-3}) with (a) potential temperature (K); (b) salt (psu); and (c) pressure (Pa), for seawater. Panel (d), shows the thermal expansion coefficient, $\beta_T = -\rho_0^{-1}(\partial\rho/\partial T)_{p,S} \text{ K}^{-1}$, for each of the two curves in panel (a).

where $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$ is the *vorticity*, discussed more in later chapters. Then, omitting viscosity, we have

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\nabla B, \quad (1.182)$$

where $B = (\phi + \Phi + \boldsymbol{v}^2/2)$ is the *Bernoulli function* for constant density flow. Consider for a moment steady flows ($\partial \boldsymbol{v}/\partial t = 0$). Streamlines are, by definition, parallel to \boldsymbol{v} everywhere, and the vector $\boldsymbol{v} \times \boldsymbol{\omega}$ is everywhere orthogonal to the streamlines, so that taking the dot product of the steady version of (1.182) with \boldsymbol{v} gives $\boldsymbol{v} \cdot \nabla B = 0$. That is, for steady flows the Bernoulli function is constant along a streamline, and $DB/Dt = 0$.

Reverting back to the time-varying case, take the dot product with \boldsymbol{v} and include the density to yield

$$\frac{1}{2} \frac{\partial \rho_0 \boldsymbol{v}^2}{\partial t} + \rho_0 \boldsymbol{v} \cdot (\boldsymbol{\omega} \times \boldsymbol{v}) = -\rho_0 \boldsymbol{v} \cdot \nabla B \quad (1.183)$$

The second term on the left-hand-side vanishes identically. Defining the kinetic energy density K , or energy per unit volume, by $K = \rho_0 \boldsymbol{v}^2/2$, (1.183) becomes an

expression for the rate of change of K ,

$$\frac{\partial K}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v} B) = 0. \quad (1.184)$$

Because Φ is time-independent this may be written

$$\frac{\partial}{\partial t} \left[\rho_0 \left(\frac{1}{2} \mathbf{v}^2 + \Phi \right) \right] + \nabla \cdot \left[\rho_0 \mathbf{v} \left(\frac{1}{2} \mathbf{v}^2 + \Phi + \phi \right) \right] = 0. \quad (1.185)$$

or

$$\frac{\partial E}{\partial t} + \nabla \cdot [\mathbf{v}(E + p)] = 0. \quad (1.186)$$

where $E = K + \rho_0 \Phi$ is the total energy density (i.e, the total energy per unit volume). This has the form of a general conservation equation in which a local change in a quantity is balanced by the divergence of its flux. However, the energy flux, $\mathbf{v}(\rho_0 \mathbf{v}^2/2 + \rho_0 \Phi + \rho_0 \phi)$, is *not* simply the velocity times the energy density $\rho_0(\mathbf{v}^2/2 + \Phi)$; there is an additional term, $\mathbf{v}p$, that represents the energy transfer occurring when work is done by the fluid against the pressure force.

Now consider a volume through which there is no mass flux, for example a domain bounded by rigid walls. The rate of change of energy within that volume is then given by the integral of (1.183)

$$\frac{d\hat{K}}{dt} \equiv \frac{d}{dt} \int_V K dV = - \int_V \nabla \cdot (\rho_0 \mathbf{v} B) dV = - \int_S \rho_0 B \mathbf{v} \cdot d\mathbf{S} = 0, \quad (1.187)$$

using the divergence theorem, and where \hat{K} is the total kinetic energy. Thus, the total kinetic energy within the volume is conserved. Note that for a constant density fluid the gravitational potential energy, \hat{P} , is given by

$$\hat{P} = \int_V \rho_0 g z dV, \quad (1.188)$$

which is a constant, not affected by a re-arrangement of the fluid. Thus, in a constant density fluid there is no exchange between kinetic energy and potential energy and the kinetic energy itself is conserved.

1.9.2 Variable density fluids

We start with the momentum equation

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \rho \nabla \Phi, \quad (1.189)$$

and take its dot product with \mathbf{v} to obtain an equation for the evolution of kinetic energy,

$$\begin{aligned} \frac{1}{2} \rho \frac{D\mathbf{v}^2}{Dt} &= -\mathbf{v} \cdot \nabla p - \rho \mathbf{v} \cdot \nabla \Phi \\ &= -\nabla \cdot (p\mathbf{v}) + p \nabla \cdot \mathbf{v} - \rho \mathbf{v} \cdot \nabla \Phi. \end{aligned} \quad (1.190)$$

From (1.90) the internal energy equation for adiabatic flow is

$$\rho \frac{DI}{Dt} = -\rho p \frac{D\alpha}{Dt} = -p \nabla \cdot \mathbf{v} \quad (1.191)$$

where the second equality follows by use of the continuity equation. Finally, and somewhat trivially, the potential energy density obeys

$$\rho \frac{D\Phi}{Dt} = \rho \mathbf{v} \cdot \nabla \Phi. \quad (1.192)$$

Adding (1.190), (1.191) and (1.192) we obtain

$$\rho \frac{D}{Dt} \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi \right) = -\nabla \cdot (p\mathbf{v}), \quad (1.193)$$

which, on expanding the material derivative and using the mass conservation equation, becomes

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi \right) \right] + \nabla \cdot \left[\rho \mathbf{v} \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi + p/\rho \right) \right] = 0. \quad (1.194)$$

This may be written

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot [\mathbf{v}(E + p)] = 0}, \quad (1.195)$$

where $E = \rho(\mathbf{v}^2/2 + I + \Phi)$ is the total energy per unit volume, or the total energy density, of the fluid. This is the energy equation for an unforced, inviscid and adiabatic, compressible fluid. Just as for the constant density case, the energy flux contains the term $p\mathbf{v}$ that represents the work done against pressure and, again, the second term vanishes when integrated over a closed domain with rigid boundaries, implying that the total energy is conserved. However, now there can now be an exchange of energy between kinetic, potential and internal components. The quantity $\sigma = I + p\alpha + \Phi = h + \Phi$ is sometimes referred to as the static energy, or the dry static energy. However, it is not a component of the globally conserved total energy; the conserved energy contains only the quantity $I + \Phi$ plus the kinetic energy, and it is only the *flux* of static energy that affects the energetics. For an ideal gas we have $\sigma = c_v T + RT + \Phi = c_p T + \Phi$, and if the potential is caused by a uniform gravitational field then $\sigma = c_p T + gz$.

Bernoullis' theorem

For steady flow $\partial/\partial t = 0$ and $\nabla \cdot \rho\mathbf{v} = 0$ so that (1.194) may be written $\mathbf{v} \cdot \nabla B = 0$ where B is the Bernoulli function given by

$$B = \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi + p/\rho \right) = \left(\frac{1}{2} \mathbf{v}^2 + h + \Phi \right). \quad (1.196)$$

This means that, for steady flow,

$$\frac{DB}{Dt} = 0, \quad (1.197)$$

and the Bernoulli function is a constant along streamline. For an ideal gas in a constant gravitation field $B = \mathbf{v}^2/2 + c_p T + gz$.

For adiabatic flow we also have $D\theta/Dt = 0$. Thus, steady flow is both along surfaces of constant θ and along surfaces of constant B , and the vector

$$\mathbf{l} = \nabla \theta \times \nabla B \quad (1.198)$$

is parallel to streamlines.¹⁰ A related result for unsteady flow is given in section 4.9.

1.9.3 Viscous Effects

We might expect that viscosity will always act to reduce the kinetic energy of a flow, and we will demonstrate this for a constant density fluid. Retaining the viscous term in (1.180), the energy equation becomes

$$\frac{d\hat{E}}{dt} \equiv \frac{d}{dt} \int_V E dV = \mu \int_V \mathbf{v} \cdot \nabla^2 \mathbf{v} dV. \quad (1.199)$$

The right hand side is negative definite. To see this we use the vector identity

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}, \quad (1.200)$$

and because $\nabla \cdot \mathbf{v} = 0$ we have $\nabla^2 \mathbf{v} = -\nabla \times \boldsymbol{\omega}$, where $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$. Thus,

$$\frac{d\hat{E}}{dt} = -\mu \int_V \mathbf{v} \cdot (\nabla \times \boldsymbol{\omega}) dV = -\mu \int_V \boldsymbol{\omega} \cdot (\nabla \times \mathbf{v}) dV = -\mu \int_V \boldsymbol{\omega}^2 dV, \quad (1.201)$$

after integrating by parts, providing $\mathbf{v} \times \boldsymbol{\omega}$ vanishes at the boundary. Thus, viscosity acts to extract kinetic energy from the flow. The loss of kinetic energy reappears as an irreversible warming of the fluid ('Joule heating'), and the total energy of the fluid is conserved, but this effect plays no role in a constant density fluid. The effect is normally locally small, at least in the earth's ocean and atmosphere, although it is sometimes included in comprehensive General Circulation Models.

1.10 AN INTRODUCTION TO NON-DIMENSIONALIZATION AND SCALING

The units we use to measure length, velocity and so on are irrelevant to the dynamics, and not necessarily the most appropriate units for a given problem. Rather, it is convenient to express the equations of motion, so far as is possible, in so-called 'nondimensional' variables, by which we mean expressing every variable (such as velocity) as the ratio of its value to some reference value. For velocity the reference could, for example, be the speed of light — but this would not be very helpful for fluid dynamical problems in the earth's atmosphere or ocean! Rather, we should choose the reference value as a natural one for a given flow, in order that, so far as possible, the nondimensional variables are order-unity quantities, and doing this is called *scaling the equations*. Evidently, there is no reference velocity that is universally appropriate, and much of the art of fluid dynamics lies in choosing sensible scaling factors for the problem at hand. Non-dimensionalization plays an important role in fluid dynamics, and we introduce it here with a simple example.

1.10.1 The Reynolds number

Consider the constant-density momentum equation in Cartesian coordinates. If a typical velocity is U , a typical length is L , a typical timescale is T , and a typical value of the pressure deviation is Φ , then the approximate sizes of the various terms in the momentum equation are given by

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \phi + \nu \nabla^2 \mathbf{v} \quad (1.202a)$$

$$\frac{U}{T} \quad \frac{U^2}{L} \sim \frac{\Phi}{L} \quad \nu \frac{U}{L^2}. \quad (1.202b)$$

The ratio of the inertial terms to the viscous terms is $(U^2/L)/(\nu U/L^2) = UL/\nu$, and this is the *Reynolds number*.¹¹ More formally, we can nondimensionalize the momentum equation by writing

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{U}, \quad \hat{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \hat{t} = \frac{t}{T}, \quad \hat{\phi} = \frac{\phi}{\Phi}, \quad (1.203)$$

where the terms with hats on are *nondimensional* values of the variables and the capitalized quantities are known as *scaling values*, and these are the approximate magnitudes of the variables. We choose the nondimensionalization so that the nondimensional variables are of order unity. Thus, for example, we choose U so that $u = \mathcal{O}(U)$ where this should be taken to mean that the magnitude of the variable u is approximately U , or that $u \sim U$, and we say that ‘ u scales like U ’. [This $\mathcal{O}()$ notation differs somewhat from the conventional mathematical meaning of ‘order’, in which $a = \mathcal{O}(\epsilon^\alpha)$ represents a limit in which $a/\epsilon^\alpha \rightarrow \text{constant}$ as $\epsilon \rightarrow 0$.] Thus, if there are well-defined length and velocity scales in the problem, and we choose these scales to perform the nondimensionalization, then the nondimensional variables are of order unity. That is, $\hat{u} = \mathcal{O}(1)$, and similarly for the other variables.

Because there are no external forces in this problem, appropriate scaling values for time and pressure are

$$T = \frac{L}{U}, \quad \Phi = U^2. \quad (1.204)$$

Substituting (1.203) and (1.204) into the momentum equation we obtain

$$\frac{U^2}{L} \left[\frac{\partial \hat{\mathbf{v}}}{\partial \hat{t}} + (\hat{\mathbf{v}} \cdot \hat{\nabla}) \hat{\mathbf{v}} \right] = -\frac{U^2}{L} \hat{\nabla} \hat{\phi} + \frac{\nu U}{L^2} \hat{\nabla}^2 \hat{\mathbf{v}}, \quad (1.205)$$

where $\hat{\nabla}$ is the nondimensional gradient operator. Eq. (1.205) simplifies to

$$\frac{\partial \hat{\mathbf{v}}}{\partial \hat{t}} + (\hat{\mathbf{v}} \cdot \hat{\nabla}) \hat{\mathbf{v}} = -\hat{\nabla} \hat{\phi} + \frac{1}{Re} \hat{\nabla}^2 \hat{\mathbf{v}}, \quad (1.206)$$

where

$$Re \equiv \frac{UL}{\nu} \quad (1.207)$$

is, again, the Reynolds number. If we have chosen our length and velocity scales sensibly — that is, if we have scaled them properly — each variable in (1.206) is order unity, with the viscous term being multiplied by $1/Re$. There are two important conclusions:

- (i) The ratio of the importance of the inertial terms to the viscous terms is given by the *Reynolds number*, defined by (1.207). In the absence of other forces, such as those due to gravity and rotation, the Reynolds number is the only non-dimensional parameter explicitly appearing in the momentum equation. Hence its value, along with the boundary conditions, controls the behaviour of the system.
- (ii) More generally, by scaling the equations of motion appropriately the parameters determining the behaviour of the system become explicit. *Scaling the equations is intelligent nondimensionalization.*

Notes

- 1 Parts of the first few chapters, and many of the problems, draw on notes prepared over the years for a graduate class at Princeton University, originally written by Steve Garner, Isaac Held and Yoshio Kurihara and also taught by Paul Kushner and myself.
- 2 Joseph-Louis Lagrange (1736–1813) was a Franco-Italian, born and raised in Turin who then lived and worked mainly in Germany and France. He made notable contributions in analysis, number theory and mechanics and was recognized as one of the greatest mathematicians of the 18th century. He laid the foundations of the calculus of variations (to wit, the ‘Lagrange multiplier’) and first formulated the principle of least action, and his monumental treatise *Mécanique Analytique* (1788) provides a unified analytic framework (it contains no diagrams, a feature virtually emulated in Whitaker’s *Analytical Dynamics*, 1927) for all Newtonian mechanics.

Leonard Euler (1707–1783), a Swiss mathematician who lived and worked for extended periods in Berlin and St. Petersburg, made important contributions in many areas of mathematics and mechanics, including the analytical treatment of algebra, the theory of equations, calculus, number theory and classical mechanics. He was the first to establish the form of the equations of motion of fluid mechanics, writing down both the field description of fluids *and* what we now call the material or advective derivative.

Truesdell (1954) points out that ‘Eulerian’ and ‘Lagrangian’ coordinates, especially the latter, are inappropriate eponyms. The so-called Eulerian description was introduced by d’Alembert in 1749 and generalized by Euler in 1752, and the so-called Lagrangian description was introduced by Euler in 1759. The modern confusion evidently stems from a monograph by Dirichlet in 1860 that credits Euler in 1757 and Lagrange in 1788 for the respective methods. One perhaps should therefore refer to the two points of view as the ‘field’ or ‘spatial’ view, and the ‘material’ view; we will also use ‘Eulerian’ because it implies a special rules for taking derivatives, but will eschew ‘Lagrangian’.

- 3 For example Batchelor (1967).
- 4 R_d and R_v are related by the molecular weights of water and dry air, M_v and M_d , so that $\alpha \equiv R_d/R_v = M_v/M_d = 0.62$. Rather than allow the gas constant to vary, meteorologists sometimes incorporate the variation of humidity into the definition of temperature, so that instead of $p = \rho R_{\text{eff}} T$ we use $p = \rho R_d T_v$, so defining the ‘virtual temperature’, T_v . It is easy to show that $T_v \approx [1 + w(\alpha^{-1} - 1)]T$. Atmospheric GCMs often use a virtual temperature.
- 5 The form of (1.64) was suggested by de Szoeke (2003). More accurately, and with more complication, the international equation of state of seawater (Unesco 1981) is an empirical equation that fits measurements to an accuracy of order 10^{-5} (see Fofonoff 1985). Somewhat simpler, more easily computable and generally accurate formulae are also available from Mellor (1991), Wright (1997), and (with particular attention to high accuracy) McDougall et al. (2002). The formulae of Wright and McDougall *et al* are of the form:

$$\rho = \frac{p + p_0}{\lambda + \alpha_0(p + p_0)}$$

where α_0 , p_0 and λ are expressed as polynomials in potential temperature and salinity, using the Gibbs function of Feistel and Hagen (1995), which is as or more accurate than the UNESCO formula. Wright’s formula used are used for Fig. 1.5 and Fig. 1.3. Bryden (1973) provides polynomial formulae for potential temperature of

seawater in terms of temperature, salinity and pressure and this is used for Fig. 1.4. In most numerical ocean models potential temperature and salinity are the prognostic thermodynamic variables and an empirical equation of state is used to compute density and potential density.

- 6 This section does not develop thermodynamics from its fundamentals; for that see, e.g., Callen (1985).
- 7 Claude-Louis-Marie-Henri Navier (1785–1836) was a French civil engineer, professor at the *École Polytechnique* and later at the *École des Ponts et Chaussée*. He was an expert in road and bridge building (he developed the theory of suspension bridges) and, relatedly, made lasting theoretical contributions to the theory of elasticity, being the first to publish a set of general equations for the dynamics of an elastic solid. In fluid mechanics, he laid down the now-called *Navier-Stokes equations*, including the viscous terms, in 1822.
- George Gabriel Stokes (1819–1903). Irish born (in Skreen, County Sligo), he held the Lucasian chair of Mathematics at Cambridge from 1849 until his retirement. As well as having a role in the development of fluid mechanics, especially through his considerations of viscous effects, Stokes worked on the dynamics of elasticity, fluorescence, the wave theory of light, and was (rather ill-advisedly in hindsight) a proponent of the idea of an ether permeating all space.
- 8 Some sources take incompressibility to mean that density is unaffected by pressure, but this alone is insufficient to guarantee that the mass conservation equation can be approximated by $\nabla \cdot \mathbf{v} = 0$.
- 9 Following de Szoeke (2003), where a more complete discussion is to be found.
- 10 These results, usually known as Bernoulli's theorem, were developed by Daniel and Johannes Bernoulli and so should be known as the Bernoullis' theorem.
- 11 Osborne Reynolds (1842–1912) was an Irish born (Belfast) physicist who was professor of engineering at Manchester University from 1868–1905. His early work was in electricity and magnetism, but he is now most famous for his work in hydrodynamics. The 'Reynolds number,' which determines the ratio of inertial to viscous forces, and the 'Reynolds stress,' which is the stress on the mean flow due to the fluctuating components, are both named after him. He was also one of the first scientists to think about the concept of group velocity.

Further Reading

There are numerous books on hydrodynamics; some of them are:

Lamb, H. 1932. *Hydrodynamics*, 6th edn.

This is a classic text in the subject, although its notation is now too dated to make it really useful as an introduction. Another very well-known text is:

Batchelor, G. K. 1967. *An Introduction to Fluid Dynamics*.

This mainly treats incompressible flow. It is rather heavy going for the true beginner, but nevertheless is a very useful reference for the fundamentals. Two other useful references are:

Tritton, D. J. 1988. *Physical Fluid Dynamics*, 2nd edn.

Kundu, P. and I. M. Cohen 1990. *Fluid Mechanics*.

Both are introductions written at the advanced undergraduate/beginning graduate level, and are easier-going than Batchelor. Kundu and Cohen's book has more material on geophysical fluid dynamics.

There are also numerous books on thermodynamics, two clear and useful ones being:

Reif, F., 1965. *Fundamentals of Statistical and Thermal Physics*,

Callen, E. B. 1985 *Thermodynamics and an Introduction to Thermostatistics*.

Reif's book has become something of a classic, and Callen provides a rather more axiomatic approach.

An introduction to thermodynamic effects in fluids, with an emphasis on fundamental properties, is provided by:

Salmon, R., 1998, *Lectures on Geophysical Fluid Dynamics*.

Problems

It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.

David Hilbert (1862-1943).

1.1 For an infinitesimal volume, informally show that

$$\frac{D}{Dt}(\rho\phi\Delta V) = \rho\Delta V \frac{D\phi}{Dt}, \quad (\text{P1.1})$$

and hence informally deduce that

$$\frac{D}{Dt} \int_V \rho\phi \, dV = \int_V \rho \frac{D\phi}{Dt} \, dV. \quad (\text{P1.2})$$

where ϕ is an arbitrary, differentiable, property of the fluid.

1.2 Show that the derivative of an integral is given by

$$\frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \phi(x, t) \, dx = \int_{x_1}^{x_2} \frac{\partial \phi}{\partial t} \, dx + \frac{dx_2}{dt} \phi(x_2, t) - \frac{dx_1}{dt} \phi(x_1, t). \quad (\text{P1.3})$$

By generalizing to three-dimensions show that the material derivative of an integral of a fluid property is given by

$$\frac{D}{Dt} \int_V \phi(\mathbf{x}, t) \, dV = \int_V \frac{\partial \phi}{\partial t} \, dV + \int_S \phi \mathbf{v} \cdot d\mathbf{S} = \int_V \left[\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{v}\phi) \right] \, dV, \quad (\text{P1.4})$$

where the surface integral (\int_S) is over the surface bounding the volume V . Hence deduce that

$$\frac{D}{Dt} \int_V \rho\phi \, dV = \int_V \rho \frac{D\phi}{Dt} \, dV. \quad (\text{P1.5})$$

1.3 Why is there no diffusion term in the mass continuity equation?

1.4 By invoking Galilean invariance we can often choose, without loss of generality, the basic state for problems in sound waves to be such that $u_0 \equiv 0$. The perturbation velocity is then certainly larger than the basic state velocity. How can we then justify ignoring the nonlinear term in the perturbation equation, as the term $u' \partial u' / \partial x$ is certainly no smaller than the linear term $u_0 \partial u' / \partial x$?

1.5 For sound waves in air of amplitude 60 dB, show that the linear approximation to the equations of motion is a good one. For example, in a one-dimensional problem, show that the term $u' \partial u' / \partial x$ is much smaller than $\partial u' / \partial t$. What amplitude of sound wave is required for the nonlinear terms to become important? Is this achieved at a rock concert?

- 1.6 Using the observed value of molecular diffusion of heat in water, estimate how long it would take for a temperature anomaly to mix from the top of the ocean to the bottom, assuming that molecular diffusion alone is responsible. Comment on whether you think the real ocean has reached equilibrium after the last ice age (which ended about 12,000 years ago).
- 1.7 Consider the following flow:

$$u = \Gamma z$$

$$v = V \sin[k(x - ct)]$$

where Γ , V , k and c are positive constants. (This is similar to the flow in the mid-latitude troposphere — an eastward flow increasing with height, with a transverse wave superimposed.) Suppose that $\Gamma z > c$ for the region of interest. Consider particles located along the $y = 0$ axis at $t = 0$, and compute their position at some later time t . Compare this with the *streamfunction* for the flow at the same time. (*Hint*: Show that the meridional particle displacement is $\eta = \psi/(u - c)$, where ψ is the streamfunction and u and c are parameters.)

- 1.8 Consider the two-dimensional flow

$$u = A(y) \sin \omega t, \quad v = A(y) \cos \omega t. \quad (\text{P1.6})$$

The time-mean of this at a fixed point flow is zero. If A is independent of y , then fluid parcels move clockwise in a circle. What is its radius? If A does depend on y , find an *approximate* expression for the average drift of a particle,

$$\lim_{t \rightarrow \infty} \frac{\mathbf{x}(\mathbf{a}, t)}{t}$$

where \mathbf{a} is a particle label and A is suitably ‘small’. Be precise about what small means.

Partial solution: Write $A(y) = A_0 + ay$ and solve

$$\frac{dy}{dt} = (A_0 + ay) \cos \omega t \quad (\text{P1.7})$$

as a polynomial function of $\sin \omega t$. Then solve

$$\frac{dx}{dt} = (A_0 + ay) \sin \omega t$$

using the solution for y . The rectification of the standing oscillation by the oscillating trajectory now appears, because $\langle \sin \omega t \sin \omega t \rangle \neq 0$. The small parameter is the relative variation of A across a particle displacement, namely

$$\frac{\delta A}{A} = \frac{(aA_0/b)}{A_0} = \frac{a}{b}.$$

That is, the shear is small compared to the frequency.

- 1.9 Suppose that a sealed, insulated container consists of two compartments, and that one of them is filled with an ideal gas and the other is a vacuum. The partition separating the compartments is removed. How does the temperature of the gas change? (Answer: It stays the same.) Reconcile this answer with the first law of thermodynamics for an ideal gas:

$$dQ = T d\eta = dI + dW = c_v dT + p d\alpha, \quad (\text{P1.8})$$

as well as with the notion that a parcel that is ascending adiabatically through the atmosphere will generally cool.

- 1.10 Show that adiabatic flow in an ideal gas satisfies $p\rho^{-\gamma}$ where $\gamma = c_p/c_v$.
- 1.11 (a) Show that for an ideal gas (1.120) is equivalent to (1.104). You may use the Maxwell relation $(\partial\alpha/\partial\eta)_p = (\partial T/\partial p)_\eta$.
 (b) Show that for an ideal gas (1.131) is equivalent to (1.103). You may use the results of part (a).
- 1.12 Show that it follows directly from the equation of state, $P = RT/\alpha$, that the internal energy of an ideal gas is a function of temperature only.
Solution: From (1.91) and $p = RT/\alpha$ we have

$$d\eta = \frac{1}{T} \left(\frac{\partial I}{\partial T} \right)_\alpha dT + \left[\frac{1}{T} \left(\frac{\partial I}{\partial \alpha} \right)_T + \frac{R}{\alpha} \right] d\alpha. \quad (\text{P1.9})$$

But, mathematically,

$$d\eta = \left(\frac{\partial \eta}{\partial T} \right)_\alpha dT + \left(\frac{\partial \eta}{\partial \alpha} \right)_T d\alpha. \quad (\text{P1.10})$$

Equating the coefficient of dT and $d\alpha$ in these two expressions gives

$$\left(\frac{\partial \eta}{\partial T} \right)_\alpha = \frac{1}{T} \left(\frac{\partial I}{\partial T} \right)_\alpha \quad \text{and} \quad \left(\frac{\partial \eta}{\partial \alpha} \right)_T = \frac{1}{T} \left(\frac{\partial I}{\partial \alpha} \right)_T + \frac{R}{\alpha}. \quad (\text{P1.11})$$

Noting that $\partial^2\eta/(\partial\alpha\partial T) = \partial^2\eta/(\partial T\partial\alpha)$ we obtain

$$\frac{\partial^2 I}{\partial \alpha \partial T} = \frac{\partial^2 I}{\partial T \partial \alpha} - \left(\frac{\partial I}{\partial \alpha} \right)_T. \quad (\text{P1.12})$$

Thus, $(\partial I/\partial \alpha)_T = 0$. Because, in general, the internal energy may be considered either a function of temperature and density or temperature and pressure, this proves that for an ideal gas the internal energy is a function *only* of temperature.

- 1.13 Show that it follows directly from the equation of state $P = RT/\alpha$, that for an ideal gas the heat capacity at constant volume, c_v , is, at most, a function of temperature.
- 1.14 Show that for an ideal gas

$$Td\eta = c_v dT + p d\alpha. \quad (\text{P1.13})$$

and that its internal energy is given by $I = \int c_v dT$.

Solution: Let us regard η as a function of T and α , where α is the specific volume $1/\rho$. Then

$$\begin{aligned} Td\eta &= T \left(\frac{\partial \eta}{\partial T} \right)_\alpha dT + T \left(\frac{\partial \eta}{\partial \alpha} \right)_T d\alpha \\ &= c_v dT + T \left(\frac{\partial \eta}{\partial \alpha} \right)_T d\alpha \end{aligned} \quad (\text{P1.14})$$

by definition of c_v . For an ideal gas the internal energy is a function of temperature alone (problem 1.12), so that using (1.74) the pressure of a fluid $p = T(\partial\eta/\partial\alpha)_I = T(\partial\eta/\partial\alpha)_T$ and (P1.14) becomes

$$Td\eta = c_v dT + p d\alpha \quad (\text{P1.15})$$

But, *in general*, the fundamental thermodynamic relation is

$$Td\eta = dI + p d\alpha. \quad (\text{P1.16})$$

The terms on the right hand side of (P1.15) are identifiable as the change in the internal energy and the work done on a fluid, and so $dI = c_v dT$. The heat capacity need not necessarily be constant, although for air it very nearly is, but it must be a function of temperature only.

- 1.15 (a) Beginning with the expression for potential temperature for an ideal gas, $\theta = T(p_R/p)^\kappa$, where $\kappa = R/c_p$, show that

$$d\theta = \frac{\theta}{T}(dT - \alpha dp), \quad (\text{P1.17})$$

and therefore that the first law of thermodynamics may be written as

$$dQ = Td\eta = c_p \frac{T}{\theta} d\theta. \quad (\text{P1.18})$$

(b) Show that (P1.18) is generally true, without making the ideal gas approximation.

- 1.16 Obtain an expression for the Gibbs function for an ideal gas in terms of pressure and temperature.
- 1.17 From (1.117) derive the conventional equation of state for an ideal gas, and obtain expressions for the heat capacities.
- 1.18 Consider an ocean at rest with known vertical profiles of potential temperature and salinity, $\theta(z)$ and $S(z)$. Suppose we also know the equation of state in the form $\rho = \rho(\theta, S, p)$. Obtain an expression for the buoyancy frequency. Check your expression by substituting the equation of state for an ideal gas and recovering a known expression for the buoyancy frequency.
- 1.19 ♦ Obtain an expression, in terms of temperature and pressure, for the potential temperature of a van der Waals gas, with equation of state $(p + a/\alpha^2)(\alpha - b) = RT$, where a and b are constants. Show that it reduces to the expression for an ideal gas in the limit $a \rightarrow 0$, $b \rightarrow 0$.
- 1.20 ♦ In seawater, why does potential temperature differ from in situ temperature? Is it because of the compressibility (with pressure) or the thermal expansion of water? A good answer will include, *inter alia*, a physical explanation of the difference in scale heights of (1.124) and (1.135) and a discussion of the lapse rate and scale heights of water that is between 0° and 4° C.

